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MATRICES

In the middle of the 19th Century, Arthur Cayley (1821-1895), an English mathematician created a new discipline of mathematics, called **matrices**. He used matrices to represent simultaneous system of equations. As of now, theory of matrices has come to stay as an important area of mathematics. The matrices are used in game theory, allocation of expenses, budgeting for by-products etc. Economists use them in social accounting, input-output tables and in the study of inter-industry economics. Matrices are extensively used in solving the simultaneous system of equations. Linear programming has its base in matrix algebra. Matrices have found applications not only in mathematics, but in other subjects like Physics, Chemistry, Engineering, Linear Programming etc.

In this lesson we will discuss different types of matrices and algebraic operations on matrices in details.

OBJECTIVES

After studying this lesson, you will be able to:

- define a matrix, order of a matrix and cite examples thereof;
- define and cite examples of various types of matrices-square, rectangular, unit, zero, diagonal, row, column matrix;
- state the conditions for equality of two matrices;
- define transpose of a matrix;
- define symmetric and skew symmetric matrices and cite examples;
- find the sum and the difference of two matrices of the same order;
- multiply a matrix by a scalar;
- state the condition for multiplication of two matrices; and
- multiply two matrices whenever possible.
- use elementary transformations
- find inverse using elementary trnsformations

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of number system
- Solution of system of linear equations

20.1 MATRICES AND THEIR REPRESENTATIONS

Suppose we wish to express that Anil has 6 pencils. We may express it as [6] or (6) with the understanding that the number inside [ ] denotes the number of pencils that Anil has. Next suppose that we want to express that Anil has 2 books and 5 pencils. We may express it as [2 5] with the understanding that the first entry inside [ ] denotes the number of books; while the second entry, the number of pencils, possessed by Anil.

Let us now consider, the case of two friends Shyam and Irfan. Shyam has 2 books, 4 notebooks and 2 pens; and Irfan has 3 books, 5 notebooks and 3 pens.

A convenient way of representing this information is in the tabular form as follows:

	Books	Notebooks	Pens
Shyam	2	4	2
Irfan	3	5	3

We can also briefly write this as follows:

$$\begin{array}{lc} & \begin{array}{ccc}\text{First Column} & \text{Second Column} & \text{Third Column}\\ \downarrow & \downarrow & \downarrow\end{array}\\ \begin{array}{l}\text{First Row}\\ \text{Second Row}\end{array} & \begin{bmatrix}2 & 4 & 2\\ 3 & 5 & 3\end{bmatrix}\end{array}$$

This representation gives the following information:

(1) The entries in the first and second rows represent the number of objects (Books, Notebooks, Pens) possessed by Shyam and Irfan, respectively

(2) The entries in the first, second and third columns represent the number of books, the number of notebooks and the number of pens, respectively.

Thus, the entry in the first row and third column represents the number of pens possessed by Shyam. Each entry in the above display can be interpreted similarly.

The above information can also be represented as

	Shyam	Irfan
Books	2	3
Notebooks	4	5
Pens	2	3

which can be expressed in three rows and two columns as given below:

$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix}$ The arrangement is called a **matrix.** Usually, we denote a matrix by a capital letter of English alphabets, i.e. A, B, X, etc. Thus, to represent the above information in the form of a matrix, we write

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix} \text{ or } \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}$$

Note: Plural of matrix is matrices.

20.1.1 Order of a Matrix Observe the following matrices (arrangement of numbers):

$$\text{(a)} \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 1 & i \\ i & 1+i \\ 1+i & 1 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 2 & 3 & 4 & 5 \\ 4 & -1 & -2 & 0 \end{pmatrix}$$

In matrix (a), there are two rows and two columns, this is called a 2 by 2 matrix or a matrix of order 2×2. This is written as 2×2 matrix. In matrix (b), there are three rows and two columns. It is a 3 by 2 matrix or a matrix of order 3×2. It is written as 3×2 matrix. The matrix (c) is a matrix of order 3×4.

Note that there may be any number of rows and any number of columns in a matrix. If there are m rows and n columns in matrix A, its order is $m \times n$ and it is read as an $m \times n$ matrix.

Use of two suffixes i and j helps in locating any particular element of a matrix. In the above $m \times n$ matrix, the element a_{ij} belongs to the ith row and jth column.

$$A = \begin{bmatrix} a_{11}\ a_{12}\ a_{13} \cdots a_{1j} & \cdots a_{1n} \\ a_{21}\ a_{22}\ a_{23} \cdots a_{2j} & \cdots a_{2n} \\ a_{31}\ a_{32}\ a_{33} \cdots a_{3j} & \cdots a_{3n} \\ a_{i1}\ a_{i2}\ a_{i3} \cdots\ a_{ij} & \cdots a_{in} \\ a_{m1}\ a_{m2}\ a_{m3} \cdots a_{mj} & \cdots a_{mn} \end{bmatrix}$$

A matrix of order $m \times n$ can also be written as

$A = [a_{ij}]$, $i = 1, 2, ..., m$; and
$j = 1, 2, ..., n$

Example 20.1 Write the order of each of the following matrices:

(i) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$ (iv) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 10 \end{pmatrix}$

Solution: The order of the matrix

(i) is 2×2 (ii) is 3×1

(iii) is 1×3 (iv) is 2×3

Example 20.2 For the following matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 3 & 2 & 5 \\ 3 & 2 & 3 & 6 \end{bmatrix}$$

(i) find the order of A

(ii) write the total number of elements in A

(iii) write the elements a_{23}, a_{32}, a_{14} and a_{34} of A

(iv) express each element 3 in A in the form a_{ij}.

Solution: The order of the matrix

(i) Since A has 3 rows and 4 columns, A is of order 3×4.

(ii) number of elements in $A = 3 \times 4 = 12$

(iii) $a_{23} = 2$; $a_{32} = 2$; $a_{14} = 4$ and $a_{34} = 6$

(iv) a_{22}, a_{31} and a_{33}

Example 20.3 If the element in the ith row and jth column of a 2×3 matrix A is given by $\dfrac{i+2j}{2}$, write the matrix A.

Solution: Here, $a_{ij} = \dfrac{i+2j}{2}$ (Given)

$$a_{11} = \frac{1+2\times1}{2} = \frac{3}{2};\ a_{12} = \frac{1+2\times2}{2} = \frac{5}{2};\ a_{13} = \frac{1+2\times3}{2} = \frac{7}{2}$$

$$a_{21} = \frac{2+2\times1}{2} = 2;\ a_{22} = \frac{2+2\times2}{2} = 3;\ a_{23} = \frac{2+2\times3}{2} = 4$$

Thus, $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 2 & 3 & 4 \end{bmatrix}$$

Example 20.4 There are two stores A and B. In store A, there are 120 shirts, 100 trousers and 50 cardigans; and in store B, there are 200 shirts, 150 trousers and 100 cardigans. Express this information in tabular form in two different ways and also in the matrix form.

Solution:

Tabular Form 1

	Shirts	Trousers	Cardigans
Store A	120	100	50
Store B	200	150	100

Matrix Form

$$\Rightarrow \begin{bmatrix} 120 & 100 & 50 \\ 200 & 150 & 100 \end{bmatrix}$$

Tabular Form 2

	Store A	Store B
Shirts	120	200
Trousers	100	150
Cardigans	50	100

Matrix Form

$$\Rightarrow \begin{bmatrix} 120 & 200 \\ 100 & 150 \\ 50 & 100 \end{bmatrix}$$

CHECK YOUR PROGRESS 20.1

1. Marks scored by two students A and B in three tests are given in the adjacent table. Represent this information in the matrix form, in two ways

	Test 1	Test 2	Test 3
A	56	65	71
B	29	37	57

2. Three firms X, Y and Z supply 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively, to a contractor. Express this information in the matrix form in two ways.

3. In family P, there are 4 men, 6 women and 3 children; and in family Q, there are 4 men, 3 women and 5 children. Express this information in the form of a matrix of order 2×3.

4. How many elements in all are there in a

(a) 2×3 matrix (b) 3×4 matrix (c) 4×2 matrix

(d) 6×2 matrix (e) $a \times b$ matrix (f) $m \times n$ matrix

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Notes

5. What are the possible orders of a matrix if it has
(a) 8 elements (b) 5 elements (c) 12 elements (d) 16 elements

6. In the matrix A,

$$A = \begin{bmatrix} 5 & 1 & 8 & 0 & 5 \\ 7 & 6 & 7 & 4 & 6 \\ 3 & 9 & 3 & -3 & 9 \\ 4 & 4 & 8 & 5 & 1 \end{bmatrix}$$

find: (a) number of rows;

(b) number of columns;

(c) the order of the matrix A;

(d) the total number of elements in the matrix A;

(e) $a_{14}, a_{23}, a_{34}, a_{45}$ and a_{33}

7. Construct a 3 $\times$ 3 matrix whose elements in the ith row and jth column is given by

(a) $i - j$ (b) $\dfrac{i^2}{j}$ (c) $\dfrac{(i+2j)^2}{2}$ (d) $3j - 2i$

8. Construct a 3×2 matrix whose elements in the ith row and jth column is given by

(a) $i + 3j$ (b) $5.i.\,j.$ (c) i^j (d) $i + j - 2$

20.2 TYPES OF MATRICES

Row Matrix : A matrix is said to be a *row matrix* if it has only one row, but may have any number of columns, e.g. the matrix $\begin{bmatrix} 1 & 6 & 0 & 1 & 2 \end{bmatrix}$ is a row matrix.

T[he order of a row matrix is 1 $\times$ n.

Column Matrix : A matrix is said to be a *column matrix* if it has only one column, but may have any number of rows, e.g. the matrix $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 7 \end{bmatrix}$ is a column matrix. The order of a column matrix is $m \times 1$

Square Matrix : A matrix is said to be a *square matrix* if number of rows is equal to the number of columns, e.g. the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 1 \\ 3 & 4 & 2 \end{bmatrix}$ having 3 rows and 3 columns is a square matrix. The order of a square matrix is $n \times n$ or simply n.

The diagonal of a square matrix from the top extreme left element to the bottom extreme right element is said to be the principal diagonal. The principal diagonal of the matrix $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ contains elements 2, 1 and 9.

Note: In any given matrix $A = [a_{ij}]$ of order $m \times n$, the elements of the principal diagonal are $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$

Rectangular Matrix : A matrix is said to be a *rectangular matrix* if the number of rows is not equal to the number of columns, e.g. the matrix $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 0 \\ -1 & 2 & 1 & 3 \end{bmatrix}$ having 3 rows and 4 columns is a rectangular matrix.It may be noted that a row matrix of order $1 \times n\ (n \neq 1)$ and a column matrix of order $m \times 1\ (m \neq 1)$ are rectangular matrix.

Zero or Null Matrix : A matrix each of whose element is zero is called a *zero or null matrix*, e.g. each of the matrix

$$[0 \quad 0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a zero matrix. Zero matrix is denoted by O.

Note: A zero matrix may be of any order $m \times n$.

Diagonal Matrix : A square matrix is said to be a *diagonal matrix*, if all elements other than those occuring in the principal diagonal are zero, i.e., if $A = [a_{ij}]$ is a square matrix of order m $\times$ n, then it is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

For example, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ are diagonal matrices.

Note: A diagonal matrix $A = [a_{ij}]_{n \times n}$ is also written as $A =$ diag $[a_{11}, a_{12}, a_{13}, \ldots, a_{nn}]$

Scalar Matrix : A diagonal matrix is said to be a *scalar matrix* if all the elements in its principal diagonal are equal to some non-zero constant, say k e.g., the matrix $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a scalar matrix.

Note: A square zero matrix is not a scalar matrix.

Unit or Identity Matrix : A scalar matrix is said to be a *unit or identity matrix*, if all of its elements in the principal diagonal are unity. It is denoted by I_n, if it is of order n e.g., the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Note: A square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = \begin{cases} 0, \text{when } i \neq j \\ 1, \text{when } i = j \end{cases}$

Equal Matrices : Two matrices are said to be equal if they are of the same order and if their corresponding elements are equal.

If A is a matrix of order $m \times n$ and B is a matrix of order $p \times r$, then $A = B$ if

(1) $m = p; n = r;$ and

(2) $a_{ij} = b_{ij}$ for all 3×2 and $j = 1, 2, 3, \ldots, n$

Two matrices X and Y given below are not equal, since they are of different orders, namely 2×3 and 3×2 respectively.

$$X = \begin{bmatrix} 7 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix}, \; Y = \begin{bmatrix} 7 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}$$

Also, the two matrices P and Q are not equal, since some elements of P are not equal to the corresponding elements of Q.

$$P = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix}, \; Q = \begin{bmatrix} -1 & 3 & 6 \\ 0 & 2 & 1 \end{bmatrix}$$

Example 20.5 Find whether the following matrices are equal or not:

(i) $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$

(ii) $P = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \end{bmatrix}, \; Q = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $X = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}$

Solution:

(i) Matrices A and B are of the same order 2×2. But some of their corresponding elements are different. Hence, $A \neq B$.

(ii) Matrices P and Q are of different orders, So, $P \neq Q$.

(iii) Matrices X and Y are of the same order 3×3, and their corresponding elements are also equal.

So, $X = Y$.

Example 20.6 Determine the values of x and y, if

(i) $\begin{bmatrix} x & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ y \end{bmatrix}$ (iii) $\begin{bmatrix} x & 2 \\ 3 & -y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

Solution: Since the two matrices are equal, their corresponding elements should be equal.

(i) $x = 2$ (ii) $x = 4, \; y = 3$ (iii) $x = 1, \; y = -5$

Example 20.7 For what values of a, b, c and d, are the following matrices equal?

(i) $A = \begin{bmatrix} a & -2 & 2b \\ 6 & 3 & d \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 4 \\ 6 & 5c & 2 \end{bmatrix}$

(ii) $P = \begin{bmatrix} a & b-2d \\ -3 & 2b \\ a+c & 7 \end{bmatrix}, \quad Q = \begin{bmatrix} 5 & 1 \\ -3 & 6 \\ 4 & 7 \end{bmatrix}$

Solution:

(i) The given matrices A and B will be equal only if their corresponding elements are equal, i.e. if

$$a = 1,\ 2b = 4,\ 3 = 5c, \text{ and } d = 2$$

$$\Rightarrow a = 1,\ b = 2,\ c = \frac{3}{5} \text{ and } d = 2$$

Thus, for $a = 1,\ b = 2,\ c = \frac{3}{5}$ and $d = 2$ matrices A and B are equal.

(ii) The given matrices P and Q will be equal if their corresponding elements are equal, i.e. if

$$2b = 6,\ b - 2d = 1,\ a = 5 \text{ and } a + c = 4$$

$$\Rightarrow a = 5,\ b = 3,\ c = -1 \text{ and } d = 1$$

Thus, for $a = 5,\ b = 3,\ c = -1$ and $d = 1$, matrices P and Q are equal.

CHECK YOUR PROGRESS 20.2

1. Which of the following matrices are

(a) row matrices (b) column matrices (c) square matrices (d) diagonal matrices (e) scalar matrices (f) identity matrices and (g) zero matrices

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 7 \\ 8 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 8 \\ 1 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G = [3 \quad 4 \quad 10 \quad 8],\ H = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 4 & 9 \end{bmatrix},\ I = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}$$

2. Find the values of a, b, c and d if

(a) $\begin{bmatrix} b & 2c \\ b+d & c-2a \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 8 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} a+2 & 4 \\ b+3 & 25 \end{bmatrix} = \begin{bmatrix} 4 & 2c \\ 6 & 5d \end{bmatrix}$

(c) $\begin{bmatrix} 2a & b \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ d & 3c \end{bmatrix}$

3. Can a matrix of order 1×2 be equal to a matrix of order 2×1?

4. Can a matrix of order 2×3 be equal to a matrix of order 3×3?

20.3 TRANSPOSE OF A MATRIX

Associated with each given matrix there exists another matrix called its *transpose*. The transpose of a given matrix A is formed by interchanging its rows and columns and is denoted by A' or A^t, e.g. if

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 3 \\ 7 & 6 & 1 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 0 & 6 \\ -3 & 3 & 1 \end{bmatrix}$$

In general, If $A = [a_{ij}]$ is an $m \times n$ matrix, then the transpose A' of A is the $n \times m$ matrix; and, (a_{ij})th element of A = (a_{ji})th element of A'

20.3.1 Symmetric Matrix

A square matrix A is said to be a *symmetric matrix* if $A' = A$.

For example,

$$If\ A = \begin{bmatrix} 2 & 3i & 1-i \\ 3i & 4 & 2i \\ 1-i & 2i & 5 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 2 & 3i & 1-i \\ 3i & 4 & 2i \\ 1-i & 2i & 5 \end{bmatrix}$$

Since $A' = A$, A is a symmetric matrix.

Note: (1) In a symmetric matrix $A = [a_{ij}]_{n \times n}$,
$a_{ij} = a_{ji}$ for all i and j

(2) A rectangular matrix can never be symmetric.

20.3.2 Skew-Symmetric Matrix

A square matrix A is said to be a *skew symmetric* if $A' = -A$, i.e. $a_{ij} = -a_{ji}$ for all i and j.

For example,

If $A = \begin{bmatrix} 0 & c & d \\ -c & 0 & f \\ -d & -f & 0 \end{bmatrix}$, *then* $A' = \begin{bmatrix} 0 & -c & -d \\ c & 0 & -f \\ d & f & 0 \end{bmatrix}$

But $A' = \begin{bmatrix} 0 & -c & -d \\ c & 0 & -f \\ d & f & 0 \end{bmatrix}$, which is the same as A'

$A' = -A$

Hence, A is a skew symmetric matrix

Note: In a skew symmetric matrix $A = \left[a_{ij}\right]_{n \times n}$, $a_{ij} = 0$, for $i = j$

i.e. all elements in the principal diagonal of a skew symmetric matrix are zeroes.

20.4 SCALAR MULTIPLICATION OF A MATRIX

Let us consider the following situation:

The marks obtained by three students in English, Hindi, and Mathematics are as follows:

	English	Hindi	Mathematics
Elizabeth	20	10	15
Usha	22	25	27
Shabnam	17	25	21

It is also given that these marks are out of 30 in each case. In matrix form, the above information can be written as

$$\begin{bmatrix} 20 & 10 & 15 \\ 22 & 25 & 27 \\ 17 & 25 & 21 \end{bmatrix}$$

(It is understood that rows correspond to the names and columns correspond to the subjects)

If the maximum marks are doubled in each case, then the marks obtained by these girls will also be doubled. In matrix form, the new marks can be given as:

$$\begin{bmatrix} 2\times 20 & 2\times 10 & 2\times 15 \\ 2\times 22 & 2\times 25 & 2\times 27 \\ 2\times 17 & 2\times 25 & 2\times 21 \end{bmatrix} \text{ which is equal to } \begin{bmatrix} 40 & 20 & 30 \\ 44 & 50 & 54 \\ 34 & 50 & 42 \end{bmatrix}$$

So, we write that

$$2\times\begin{bmatrix} 20 & 10 & 15 \\ 22 & 25 & 27 \\ 17 & 25 & 21 \end{bmatrix} = \begin{bmatrix} 2\times 20 & 2\times 10 & 2\times 15 \\ 2\times 22 & 2\times 25 & 2\times 27 \\ 2\times 17 & 2\times 25 & 2\times 21 \end{bmatrix} = \begin{bmatrix} 40 & 20 & 30 \\ 44 & 50 & 54 \\ 34 & 50 & 42 \end{bmatrix}$$

Now consider another matrix

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{bmatrix}$$

Let us see what happens, when we multiply the matrix A by 5

$$\text{i.e.}\quad 5\times A = 5A = 5\times\begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5\times 3 & 5\times 2 \\ 5\times(-2) & 5\times 0 \\ 5\times 1 & 5\times 6 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ -10 & 0 \\ 5 & 30 \end{bmatrix}$$

When *a* matrix is multiplied by a scalar, then each of its element is multiplied by the same scalar.

For example,

$$\text{if } A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix} \text{ then } kA = \begin{bmatrix} k\times 2 & k\times(-1) \\ k\times 6 & k\times 3 \end{bmatrix} = \begin{bmatrix} 2k & -k \\ 6k & 3k \end{bmatrix}$$

When $k = -1$, $kA = (-1)A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$

So, $(-1)A = -A$

Thus, if $A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$

Example 20.8 If $A = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix}$, find

(i) $2A$ (ii) $\frac{1}{2}A$ (iii) $-A$ (iv) $\frac{2}{3}A$

Solution:

(i) Here, $2A = 2 \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\times(-2) & 2\times 3 & 2\times 4 \\ 2\times(-1) & 2\times 0 & 2\times 1 \end{bmatrix} = \begin{bmatrix} -4 & 6 & 8 \\ -2 & 0 & 2 \end{bmatrix}$

(ii) $$\frac{1}{2}A = \frac{1}{2} \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times(-2) & \frac{1}{2}\times 3 & \frac{1}{2}\times 4 \\ \frac{1}{2}\times(-1) & \frac{1}{2}\times 0 & \frac{1}{2}\times 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} & 2 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(iii) $$-A = (-1)\times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -1 \end{bmatrix}$$

iv) $$\frac{2}{3}A = \frac{2}{3} \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 & \frac{8}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

CHECK YOUR PROGRESS 20.3

1. If $A = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$, find:

   (a) $4A$ (b) $-A$ (c) $\frac{1}{2}A$ (d) $-\frac{3}{2}A$

2. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, find:

   (a) $5A$ (b) $-3A$ (c) $\frac{1}{3}A$ (d) $-\frac{1}{2}A$

3. If $A = \begin{bmatrix} -1 & 0 \\ 4 & 2 \\ 0 & -1 \end{bmatrix}$, find $(-7)A$

4. If $X = \begin{pmatrix} 3 & 0 & 1 \\ 4 & -2 & 0 \\ -1 & 0 & 5 \end{pmatrix}$, find:

(a) $5X$ (b) $-4X$ (c) $\frac{1}{3}X$ (d) $-\frac{1}{2}X$

5. Find A' (transpose of A):

(a) $A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ (b) $A = \begin{pmatrix} 4 & 10 & 9 \\ 6 & 8 & 7 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & -2 \\ 4 & -1 \\ -6 & 9 \end{pmatrix}$ (d) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. For any matrix A, prove that $(A')' = A$

7. Show that each of the following matrices is a symmetric matrix:

(a) $\begin{bmatrix} 2 & -4 \\ -4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & -3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. Show that each of the following matrices is a skew symmetric matrix:

(a) $\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$ (b) $\begin{bmatrix} 0 & i & 4 \\ -i & 0 & 2-i \\ -4 & -2+i & 0 \end{bmatrix}$

Notes

(c) $\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 5 \\ -7 & -5 & 0 \end{bmatrix}$

20.5 ADDITION OF MATRICES

Two students A and B compare their performances in two tests in Mathematics, Physics and English. The maximum marks in each test in each subject are 50. The marks scored by them are as follows:

First Test

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 50 & 38 & 33 \\ 47 & 40 & 36 \end{bmatrix} \end{array}$$

Second Test

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 45 & 32 & 30 \\ 42 & 30 & 39 \end{bmatrix} \end{array}$$

How can we find their total marks in each subject in the two tests taken together ?

Observe that the new matrix giving the combined information of two matrices

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 50+45 & 38+32 & 33+30 \\ 47+42 & 40+30 & 36+39 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 95 & 70 & 63 \\ 89 & 70 & 75 \end{bmatrix} \end{array}$$

This new matrix is called the *sum* of the given matrices.

If *A* and *B* are any two given matrices of the same order, then their sum is defined to be a matrix *C* whose respective elements are the sum of the corresponding elements of the matrices *A* and *B* and we write this as $C = A + B$.

1. The order of the matrix C will also be the same as that of A and B.
2. It is not possible to add two matrices of different orders.

Example 20.9 If 2×2 and $B = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}$, then find $A + B$.

Solution: Since the given matrices A and B are of the same order, i.e. 2×2, we can add them. So,

$$A + B = \begin{bmatrix} 1+5 & 3+2 \\ 4+1 & 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 \\ 5 & 2 \end{bmatrix}$$

Example 20.10 If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 4 \\ 1 & 2 & 1 \end{bmatrix}$, then find $A + B$.

Solution: Since the given matrices A and B are of the same order, i.e. 2×2, we can add them. So,

$$A + B = \begin{bmatrix} 0+3 & 1+0 & -1+4 \\ 2+1 & 3+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}$$

20.5.1 Properties of Addition

Recall that in case of numbers, we have

(i) $x + y = y + x$, i.e., addition is commutative

(ii) $x + (y + z) = (x + y) + z$, i.e., addition is associative

(iii) $x + 0 = x$, i.e., additive identity exists

(iv) $x + (-x) = 0$, i.e., additive inverse exists

Let us now find if these properties hold true in case of matrices too:

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$, Then,

$$A + B = \begin{bmatrix} 1+0 & 2-2 \\ -1+1 & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$B + A = \begin{bmatrix} 0+1 & -2+2 \\ 1+(-1) & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

We see that $A + B$ and $B + A$ denote the same matrix. Thus, in general,

For any two matrices A and B of the same order, $A + B = B + A$

i.e. matrix addition is commutative

Let $A = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$. Then,

$$A+(B+C)=\begin{bmatrix}0 & 3\\ -2 & 1\end{bmatrix}+\begin{bmatrix}1+1 & -4+0\\ 0+2 & 2+3\end{bmatrix}=\begin{bmatrix}0 & 3\\ -2 & 1\end{bmatrix}+\begin{bmatrix}2 & -4\\ 2 & 5\end{bmatrix}$$

$$=\begin{bmatrix}0+2 & 3+(-4)\\ -2+2 & 1+5\end{bmatrix}= =\begin{bmatrix}2 & -1\\ 0 & 6\end{bmatrix}$$

and $$(A+B)+C=\begin{bmatrix}0+1 & 3+(-4)\\ -2+0 & 1+2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}=\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}+\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+1 & -1+0\\ -2+2 & 3+3\end{bmatrix}=\begin{bmatrix}2 & -1\\ 0 & 6\end{bmatrix}$$

We see that $A + (B + C)$ and $(A + B) + C$ denote the same matrix. Thus, in general

For any three matrices A, B and C of the same order,

$A + (B + C) = (A + B) + C$ i.e., matrix addition is associative.

Recall that we have talked about zero matrix. A zero matrix is that matrix, all of whose elements are zeroes. It can be of any order.

Let $A=\begin{bmatrix}2 & -2\\ 4 & 5\end{bmatrix}$ and $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$. Then,

$$A+O=\begin{bmatrix}2+0 & -2+0\\ 4+0 & 5+0\end{bmatrix}=\begin{bmatrix}2 & -2\\ 4 & 5\end{bmatrix}=A$$

and $$O+A=\begin{bmatrix}0+2 & 0-2\\ 0+4 & 0+5\end{bmatrix}=\begin{bmatrix}2 & -2\\ 4 & 5\end{bmatrix}=A$$

We see that $A + O$ and $O + A$ denote the same matrix A.

Thus, we find that $A + O = A = O + A$, where O is a zero matrix.

The matrix O, which is a zero matrix, is called the additive identity.

Additive identity is a zero matrix, which when added to a given matrix, gives the same given matrix, i.e., $A + O = A = O + A$.

Example 20.11 If $A=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}$, $B=\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}$ and $C=\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, then

find: (a) $A + B$ (b) $B + C$ (c) $(A + B) + C$ (d) $A + (B + C)$

Solution:

(a) $$A+B=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}2+(-3) & 0+1\\ 1+1 & 3+2\end{bmatrix}=\begin{bmatrix}-1 & 1\\ 2 & 5\end{bmatrix}$$

(b) $$B+C=\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}(-3)+(-1) & 1+0\\ 1+0 & 2+3\end{bmatrix}=\begin{bmatrix}-4 & 1\\ 1 & 5\end{bmatrix}$$

(c) $$(A+B)+C=\begin{bmatrix}-1 & 1\\ 2 & 5\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$ ... [From (a)]

$$= \quad =\begin{bmatrix}(-1)+(-1) & 1+0\\ 2+0 & 5+3\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 2 & 8\end{bmatrix}$$

(d) $$A+(B+C)=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-4 & 1\\ 1 & 5\end{bmatrix}$$ ... [From (b)]

$$=\begin{bmatrix}2+(-4) & 0+1\\ 1+1 & 3+5\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 2 & 8\end{bmatrix}$$

Example 20.12 If $A=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$ and $O=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

then find (a) $A+O$ (b) $O+A$

What do you observe?

Solution: (a) $$A+O=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-2+0 & 3+0 & 5+0\\ 1+0 & -1+0 & 0+0\end{bmatrix}=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$$

(b) $$O+A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0+(-2) & 0+3 & 0+5\\ 0+1 & 0+(-1) & 0+0\end{bmatrix}=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$$

From (a) and (b), we see that

$A+O=O+A=A$

20.6 SUBTRACTION OF MATRICES

Let A and B two matrices of the same order. Then the matrix $A-B$ is defined as the subtraction of B from A. $A-B$ is obtained by subtracting corresponding elements of B from the corresponding elements of A.

We can write $A-B = A+(-B)$

Note : $A-B$ and $B-A$ do not denote the same matrix, except when $A = B$.

Example 20.13 If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ then find

(a) $A-B$ (b) $B-A$

Solution : (a) We know that

$$A-B = A+(-B)$$

Since $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, we have $-B = \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$

Substituting it in (i), we get

$$A-B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 0+(-2) \\ 2+(-1) & (-1)+(-4) \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

(b) Similarly,

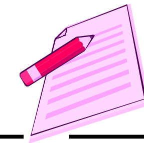
$$B-A = B+(-A)$$

$$B-A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3+(-1) & 2+0 \\ 1+(-2) & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$

Remarks : To obtain $A-B$, we can subtract directly the elements of B from the corresponding elements of A. Thus,

$$A-B = \begin{bmatrix} 1-3 & 0-2 \\ 2-1 & -1-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

and $$B-A = \begin{bmatrix} 3-1 & 2-0 \\ 1-2 & 4-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$



Example 20.14 If $A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$; $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A+B = O$, find B.

Solution : Here, it is given that A+B = O

$$\therefore \quad \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2+a & 3+b \\ -1+c & 4+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} =$$

$$\Rightarrow \quad 2+a = 0 \quad ; \quad 3+b = 0$$

$$-1+c = 0 \quad ; \quad 4+d = 0$$

$$\Rightarrow \quad a = -2; \quad b = -3; \quad c = 1 \text{ and } d = -4$$

$$\therefore \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 1 & -2 \end{pmatrix}$$

In general, given a matrix *A*, there exists another matrix *B* = (− 1) A such that *A* + *B*= *O*, then such a matrix *B* is called the additive inverse of the matrix of *A*.

CHECK YOUR PROGRESS 20.4

1. If $A = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$ then find :

   (a) $A+B$ (b) $2A+B$ (c) $A+3B$ (d) $2A+3B$

2. If $P = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 1 & -5 \end{bmatrix}$ and $Q = \begin{pmatrix} 1 & 2 & -3 \\ 4 & 1 & -5 \end{pmatrix}$, then find :

   (a) P−Q (b) Q−P (c) P− 2Q (d) 2Q−3P

3. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -1 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -4 & 0 \\ 1 & 6 & 1 \\ 2 & 0 & 7 \end{bmatrix}$, then find:

(a) $A+B$ (b) $A-B$ (c) $-A+B$ (d) $3A+2B$

4. If $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$, find the zero matrix O satisfying A+O = A.

5. If $A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \\ -4 & 0 & 1 \end{bmatrix}$ then find :

(a) $-A$ (b) $A + (-A)$ (c) $(-A)+A$

6. If $A = \begin{bmatrix} 1 & 9 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 7 & 9 \end{bmatrix}$, then find :

(a) 2A (b) 3B (c) 2A+3B (d) If 2A+3B+5X=O, what is X ?

7. If $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -4 & 0 \end{bmatrix}$, then find :

(a) A' (b) B' (c) $A+B$ (d) $(A+B)'$ (e) $A'+B'$

What do you observe ?

8. If $A = \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, then find :

(a) A−B (b) B−C (c) A−C (d) 3B−2C (e) A−B−C (f) 2A−B−3C

20.7 MULTIPLICATION OF MATRICES

Salina and Rakhi are two friends. Salina wants to buy 17 kg wheat, 3 kg pulses and 250gm ghee; while Rakhi wants to buy 15 kg wheat, 2kg pulses and 500 gm ghee. The prices of wheat, pulses and ghee per kg respectively are Rs. 8.00, Rs. 27.00 and Rs. 90.00.How much money will each spend? Clearly, the money needed by Salina and Rakhi will be :

Salina

Cost of 17 kg wheat	$\Rightarrow 17 \times$ Rs. 8	= Rs. 136.00
Cost of 3 kg pulses	$\Rightarrow 3 \times$ Rs. 27	= Rs. 81.00
Cost of 250 gm ghee	$\Rightarrow \frac{1}{4} \times$ Rs. 90	= Rs. 22.50
	Total	= Rs. 239.50

Rakhi

Cost of 15 kg wheat $\Rightarrow 15 \times$ Rs. 8 $=$ Rs. 120.00

Cost of 2 kg pulses $\Rightarrow 2 \times$ Rs. 27 $=$ Rs. 54.00

Cost of 500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 90 $=$ Rs. 45.00

Total $=$ Rs. 219.00

In matrix form, the above information can be represented as follows:

Requirements Price Money Needed

$$\begin{array}{ccc} \text{Wheat} & \text{pulses} & \text{ghee} \end{array}$$
$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 8 \\ 27 \\ 90 \end{bmatrix} = \begin{bmatrix} 17 \times 8 + 3 \times 27 + 0.250 \times 90 \\ 15 \times 8 + 2 \times 27 + 0.500 \times 90 \end{bmatrix} = \begin{bmatrix} 239.50 \\ 219.00 \end{bmatrix}$$

Another shop in the same locality quotes the following prices.

Wheat : Rs. 9 per kg.; pulses : Rs.26 per kg; ghee : Rs. 100 per kg.

The money needed by Salina and Rakhi to buy the required quantity of articles from this shop will be

Salina

17 kg wheat $\Rightarrow 17 \times$ Rs. 9 $=$ Rs. 153.00

3 kg pulses $\Rightarrow 3 \times$ Rs. 26 $=$ Rs. 78.00

250 gm ghee $\Rightarrow \frac{1}{4} \times$ Rs. 100 $=$ Rs. 25.00

Total $=$ Rs. 256.00

Rakhi

15 kg wheat $\Rightarrow 15 \times$ Rs. 9 $=$ Rs. 135.00

2 kg pulses $\Rightarrow 2 \times$ Rs. 26 $=$ Rs. 52.00

500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 100$=$ Rs. 50.00

Total $=$ Rs. 237.00

In matrix form, the above information can be written as follows :

Requirements Price Money needed

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 9.00 \\ 26.00 \\ 100.00 \end{bmatrix} = \begin{bmatrix} 17 \times 9.00 + 3 \times 26.00 + 0.250 \times 100 \\ 15 \times 900 + 2 \times 26.00 + 0.500 \times 100 \end{bmatrix} = \begin{bmatrix} 256.00 \\ 237.00 \end{bmatrix}$$

To have a comparative study, the two information can be combined in the following way:

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 27 & 26 \\ 90 & 100 \end{bmatrix} = \begin{bmatrix} 239.50 & 256.00 \\ 219.00 & 237.00 \end{bmatrix}$$

Let us see how and when we write this product :

i) The three elements of first row of the first matrix are multiplied respectively by the corresponding elements of the first column of the second matrix and added to give element of the first row and the first column of the product matrix. In the same way, the product of the elements of the second row of the first matrix to the corresponding elements of the first column of the second matrix on being added gives the element of the second row and the first column of the product matrix; and so on.

ii) The number of column of the first matrix is equal to the number of rows of the second matrix so that the first matrix is compatible for multiplication with the second matrix.

Thus, If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$, then

$$A \times B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \times \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \end{bmatrix}$$

Definition : If A and B are two matrices of order $m \times p$ and $p \times n$ respectively, then their product will be a matrix C of order $m \times n$; and if a_{ij}, b_{ij} and c_{ij} are the elements of the ith row and jth column of the matrices A, B and C respectively, then

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Example 20.15 If $A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, then find:

(a) AB (b) BA

Is $AB = BA$?

Solution : Order of A is 1×3

Order of B is 3×1

$\therefore$ Number of columns of A = Number of rows of B

$\therefore$ AB exists

Now, $AB = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

$$= \left[1\times(-2)+(-1)\times 0+2\times 2\right] = \left[-2+0+4\right] = \left[2\right]$$

Thus, $AB = [2]$, a matix of order 1×1

Again, number of columns of B = number of rows of A.

$\therefore$ BA exists

Now,

$$BA = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2\times 1 & (-2)\times(-1) & (-2)\times 2 \\ 0\times 1 & 0\times(-1) & 0\times 2 \\ 2\times 1 & 2\times(-1) & 2\times 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$$

Thus, $BA = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$, a matrix of order 3×3

From the above, we find that $AB \neq BA$

Example 20.16 Find AB and BA, if possible for the matrices A and B:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Solution : Here, Number of columns of A $\neq$ Number of rows of B

$\therefore$ AB does not exist.

Further, Number of columns of B $\neq$ Number of rows of A

$\therefore$ BA does not exist.

Example 20.17 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, then find AB and BA. Also find if AB=BA.

Solution : Here, Number of columns of A = Number of rows of B

$\therefore$ AB exists.

Further, Number of columns of B = Number of rows of A

$\therefore$ BA also exists.

$$\text{Now, } AB = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 2 + 2\times 2 & 1\times 1 + 2\times 2 \\ -1\times 2 + 0\times 2 & -1\times 1 + 0\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+4 & 1+4 \\ -2+0 & -1+0 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ -2 & -1 \end{pmatrix}_{2\times 2}$$

$$\text{and } BA = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times 1 + 1\times(-1) & 2\times 2 + 1\times 0 \\ 2\times 1 + 2\times(-1) & 2\times 2 + 2\times 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2-1 & 4+0 \\ 2-2 & 4+0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 4 \end{pmatrix}_{2\times 2}$$

Thus, $AB \neq BA$

Remarks : We observe that AB and BA are of the same order 2×2, but still $AB \neq BA$.

Example 20.18 If $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$, find AB and BA. Is $AB = BA$?

Solution : Here, both A and B are of order 2×2. So, both AB and BA exist. Now

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}_{2\times 2}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}_{2\times 2} \text{ and}$$

$$BA = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}_{2\times 2}$$

Here, both AB and BA are of the same order and $AB = BA$.

Hence, if two matrcies A and B are multiplied, then the following five cases arise:

(i) Both AB and BA exist, but are of different orders

(ii) Only one of the products AB or BA exists.

(iii) Neither AB nor BA exist.

(iv) Both AB and BA exist and are of the same order, but $AB \neq BA$.

(v) Both AB and BA exist and are of the same order. Also, $AB = BA$.

Example 20.19 If $A= A=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$ and $I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, verify that $A^2-2A-3I=O$

Solution: Here,

$$A^2=AA = \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix} = \begin{bmatrix}9+0 & 0+0\\ 0+0 & 0+9\end{bmatrix} = \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}$$

$$2A = 2\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix} = \begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}$$

and $\quad 3I = 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

$$A^2-2A-3I = -\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix} - \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix} - \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

$$= \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix} - \left\{\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix} + \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}\right\}$$

$$= \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix} - \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}$$

$$= \begin{bmatrix}9-9 & 0-0\\ 0-0 & 9-9\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = O$$

Hence, verified.

Example 20.20 Solve the matrix equation :

$$\begin{bmatrix}2 & -3\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1\\ 3\end{bmatrix}$$

Solution : Here,

$$\text{L.H.S.} = \begin{bmatrix}2 & -3\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}2x-3y\\ x+y\end{bmatrix} \Rightarrow \begin{bmatrix}2x-3y\\ x+y\end{bmatrix} = \begin{bmatrix}1\\ 3\end{bmatrix}$$

$$\Rightarrow 2x-3y=1;\ x+y=3$$

Solving these equations, we get

$x=2$ and $y=1$

Example 20.21 If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$, then find AB

Soution : Here,

$$A-B=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}-\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}1\times(-1)+1\times 1 & 1\times 1+(-1)\\ 1\times(-1)+1\times 1 & 1\times 1+(-1)\end{bmatrix}$$

$$=\begin{bmatrix}-1+1 & 1-1\\ -1+1 & 1-1\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$$

Hence, we conclude that the product of two non-zero matrices can be a zero matrix, whereas in numbers, the product of two non-zero numbers is always non-zero.

Example 20.22 For $A=\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}$, $B=\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}$ and $C=\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, find

(a) $(AB)C$ (b) $A(BC)$

Is $(AB)C=A(BC)$?

Solution : (a) $(AB)C=\left\{\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\right\}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$

$$=\begin{bmatrix}4+2 & 0-4\\ 12-5 & 0+10\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6 & -4\\ 7 & 10\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-6+0 & 0-12\\ -7+0 & 0+30\end{bmatrix}=\begin{bmatrix}-6 & -12\\ -7 & 30\end{bmatrix}$$

(b) $A(BC)=\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}\left\{\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}\right\}$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -4+0 & 0+0 \\ 1+0 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4-2 & 0-12 \\ -12+5 & 0+30 \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ -7 & 30 \end{bmatrix}$$

From (a) and (b), we find that $(AB)C = A(BC)$, i.e., matrix multiplication is associative.

CHECK YOUR PROGRESS 20.5

1. If $A = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$, find AB and BA. Is $AB = BA$?

2. If A= $B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}$, find AB and BA. Is AB = BA?

3. If A = $\begin{bmatrix} a \\ b \end{bmatrix}$ and B= $\begin{bmatrix} x & y & z \end{bmatrix}$, find AB and BA, whichever exists.

4. If A = $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$, find BA. Does AB exists?

5. If A= $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

   (a) Does AB exist? Why? (b) Does BA exist? Why?

6. If A = $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}$, find AB and BA. Is AB=BA?

MODULE - VI
Algebra -II

Notes

7. If A = $\begin{pmatrix} 1 & 2 & 0 \\ -3 & 5 & 4 \\ 5 & 3 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find AB and BA. Is AB=BA?

8. If A= $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, find AB and BA. Is AB=BA?

9. Find the values of *x* and *y* if

(a) $\begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

10. For $A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$, verify that AB=O

11. For A= $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, verify that $A^2 - 5A + I = O$, where I is a unit matrix of order 2.

12. If A = $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, B= $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ and C = $\begin{bmatrix} 4 & -3 \\ -2 & 3 \end{bmatrix}$, find :

(a) A(BC) (b) (AB)C (c) (A+B)C

(d) AC+BC (e) $A^2 - B^2$ (f) (A−B) (A+B)

13. If A = $\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, B= $\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$ and C = $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, find : (a) AC (b) BC

Is AC = BC ? What do you conclude?

14. If A = $\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$, B = $\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ and C= $\begin{pmatrix} 3 & 8 \\ 7 & -1 \end{pmatrix}$, find :

(a) B+C (b) A (B+C) (c) AB (d) AC (e) AB+AC

What do you observe?

15. For matices A = $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}$, verify that $(AB)' = B'A'$

16. If $A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, find X such that AX = B.

17. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, show that , $A^2-(a+d)A = (bc-ad)I$

18 If $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, is it true that

(a) $(A+B)^2 = A^2+B^2+2AB$? (b) $(A-B)^2 = A^2+B^2-2AB$?

(c) $(A+B)(A-B) = A^2-B^2$?

20.8 INVERTIBLE MATRICES

Definition : A square matrix of order n is invertible if there exists a square matrix B of the same order such that

$AB = I_n = BA$, Where I_n is identify matrix of order n.

In such a case, we say that the inverse of A is B and we write, $A^{-1} = B$.

Theorem 1 : Every invertible matrix possesses a unique inverse.

Proof : Let A be an invertible matrix of order

Let B and C be two inverses of A.

Then,

$$AB = BA = I_n \qquad ...(1)$$

and

$$AC = CA = I_n \qquad ...(2)$$

Now, $AB = I_n$

$\Rightarrow$ $C(AB) = C\,I_n$ [Pre-multiplying by C]

$\Rightarrow$ $(CA)\,B = C\,I_n$ [by associativity]

$\Rightarrow$ $I_n B = C\,I_n$ ($\because$ $CA = I_n$ from (ii)]

$\Rightarrow$ $B = C$ [$\because$ $I_n B = B$, $C\,I_n = C$]

Hence, an invertible matrix possesses a unique inverse.

CORROLLARY **If A is an invertible matrix then $(A^{-1})^{-1} = A$**

Proof : We have, $A\,A^{-1} = I = A^{-1}A$

$\Rightarrow$ A is the inverse of A^{-1} i.e., $A = (A^{-1})^{-1}$

Theorem 2 : A square matrix is invertible iff it is non-singular.

Proof : Let A be an invertible matrix. Then, there exists a matrix B such that

$$AB = I_n = BA$$

$\Rightarrow \quad |AB| = |I_n|$

$\Rightarrow \quad |A|\ |B| = 1$

$[\because\ |AB| = |A|\ |B|]$

$\Rightarrow \quad |A| \neq 0$

$\Rightarrow$ A is a non-singular matrix.

Conversely, let A be a non-singular square matrix of order n, then,

$$\Rightarrow \quad A\left(\frac{1}{|A|} adj\ A\right) = I_n = \left(\frac{1}{|A|} adj\ A\right) A \quad \left[\because |A| \neq 0 \therefore \frac{1}{|A|} \text{exists}\right]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} adj\ A \qquad \text{[By def. of inverse]}$$

Hence, A is an invertible matrix.

Remark : This theorem provides us a formula for finding the inverse of a non-singular square matrix.

The inverse of A is given by

$$A^{-1} = \frac{1}{|A|} adj\ A$$

20.9 ELEMENTARY TRANSFORMATIONS OR ELEMENTARY OPERATIONS OF A MATRIX

The following three operations applied on the rows (columns) of a matrix are called elementary row (column) transformations.

(i) Interchange of any two rows (columns)

If i^{th} row (column) of a matrix is interchanged with the jth row (column), it is dennoted by $R_i \leftrightarrow R_j$ or $(C_i \leftrightarrow C_j)$.

for example, $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, then by applying $R_2 \leftrightarrow R_3$

we get $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

(ii) Multiplying all elements of any row (column) of a matrix by a non-zero scalar

If the elements of ith row (column) are multiplied by a non-zero scalar k, it is denoted by $R_i \to k\ R_i\ [C_i \to k\ C_i]$

For example

If $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, then by applying $R_1 \to 2R_1$ we get $B = \begin{bmatrix} 6 & 4 & -2 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$

(iii) Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar *k*

If k times the elements of jth row (column) are added to the corresponding elements of the ith row (column), it is denoted by $R_i \to R_i + kR_j\ (C_i \to C_i + k\ C_j)$.

If $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$, then the application of elementary operation $R_3 \to R_3 + 2R_1$, gives the matrix

$$B = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 4 & 3 & 9 & 3 \end{bmatrix}$$

20.9.1 INVERSE OF A MATRIX BY ELEMENTARY OPERATIONS

We can find the inverse of a matrix, if it exists, by using either elementary row operations or column operations but not both simultaneously.

Let A be an invertible square matrix of order n, if we want to find A^{-1} by using elementary raw operations then we write

$$A = I_n\ A \qquad \text{...(i)}$$

As an elementary row operation on the product of two matrices can be affected by subjecting the pre factor to the same elementary row operation, we shall use elementary row operations on (i) so that its L.H.S reduces to In and R.H.S (after applying corresponding elementary row operations on the prefactor I_n), we get

$$I_n = BA \qquad \text{...(ii)}$$

Which means matrix B and matrix A are inverse of each other i.e. $A^{-1} = B$

Similarly if we want to find A^{-1} by using elementary column operations, we write

$$A = A\,I_n \qquad \text{...(iii)}$$

Now use elementary column operations on (iii) so that its L.H.S reduces to I_n and R.H.S (after applying corresponding elementary column operations on the post factor I_n) takes the shape

$$I_n = AB$$

$$\text{Then} \quad A^{-1} = B$$

The method is explained below with the help of some examples.

Example 20.23 Find the inverse of matrix A, using elementary column operations where,

$$A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

Solution : Writing

$$A = A\,I_2 \quad \Rightarrow \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{2} & 3 \\ 0 & 1 \end{bmatrix} \quad \text{Operating } C_2 \to C_2 + 3C_1$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A\begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{Operating } C_1 \to \frac{1}{2}C_1$$

$$\Rightarrow \quad I_2 = AB, \text{ where } B = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{Operating } C_1 \to C_1 - \frac{1}{2}C_2$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Example 20.24 Find the inverse of the matrix A using elementary row operations, where

$$A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

Solution : Writing

$$A = I_2 A$$

$$\Rightarrow \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to \frac{1}{10} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to R_2 + 5 R_1,$$

As the matrix in L.H.S contain, a row in which all elements are 0. So inverse of this matrix does not exist. Because in such case the matrix in L.H.S can not be conversed into a unit matrix.

Example 20.25 Find the inverse of the matrix A, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

Solution : We have

$$A = I\, A$$

$$\text{or} \quad \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 - R_2,$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to \frac{1}{2} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ -5 & 6 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 + R_2,\ R_3 \to R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} A \quad \text{Operating } R_3 \to \frac{1}{4} R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 + \frac{1}{2} R_3,\ R_2 \to R_2 - \frac{1}{2} R_3$$

Hence $A^{-1} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$

CHECK YOUR PROGESS 20.6

1. Find inverse of the following matrices using elementary operations :

(a) $\begin{bmatrix} 7 & 1 \\ 4 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 10 \\ 3 & 6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

LET US SUM UP

- A rectangular array of numbers, arranged in the form of rows and columns is called a *matrix*. Each number is called an *element* of the matrix.
- The *order* of a matrix having 'm' rows and 'n' columns is $m \times n$.
- If the number of rows is equal to the number of columns in a matrix, it is called a square matrix.
- A diagonal matrix is a square matrix in which all the elements, except those on the diagonal, are zeroes.
- A unit matrix of any order is a diagonal matrix of that order whose all the diagonal elements are 1.
- Zero matrix is a matrix whose all the elements are zeroes.
- Two matrices are said to be equal if they are of the same order and their corresponding elements are equal.
- A transpose of a matrix is obtained by interchanging its rows and columns.
- Matrix A is said to be symmetric if $A' = A$ and skew symmetric if $A' = -A$.
- Scalar multiple of a matrix is obtained by multiplying each elements of the matrix by the scalar.
- The sum of two matrices (of the same order) is a matrix obtained by adding corresponding elements of the given matrices.
- Difference of two matrices A and B is nothing but the sum of matrix A and the negative of matrix B.
- Product of two matrices A of order $m \times n$ and B of order $n \times p$ is a matrix of order $m \times p$, whose elements can be obtained by multiplying the rows of A with the columns of B element wise and then taking their sum.
- Product of a matrix and its inverse is equal to identity matrix of same order.
- Inverse of a matrix is always unique.
- All matrices are not necessarily invertible.
- Three points are collinear if the area of the triangle formed by these three points is zero.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=xZBbfLLfVV4

http://www.youtube.com/watch?v=ArcrdMkEmKo

http://www.youtube.com/watch?v=S4n-tQZnU6o

http://www.youtube.com/watch?v=obts_JDS6_Q

http://www.youtube.com/watch?v=01c12NaUQDw

http://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=sys

TERMINAL EXERCISE

1. How many elements are there in a matrix of order

   (a) 2×1 (b) 3×2 (c) 3×3 (d) 3×4

2. Construct a matrix of order 3×2 whose elements a_{ij} are given by

   (a) $a_{ij} = i - 2j$ (b) $a_{ij} = 3i - j$ (c) $a_{ij} = i + \frac{3}{2}j$

3. What is the order of the matrix?

   (a) $A = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & -1 & 5 \\ 7 & 6 & 1 \end{bmatrix}$

4. Find the value of x, y and z if

   (a) $\begin{bmatrix} x & y \\ z & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} x+y & z \\ 6 & x-y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 6 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} x-2 & 3 \\ 0 & y+5 \end{bmatrix} = \begin{bmatrix} 1 & z \\ y+z & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & -2 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & 4 \end{bmatrix}$, find :

   (a) A+B   (b) 2A   (c) 2A−B

6. Find X, if

   (a) $\begin{bmatrix} 4 & 5 \\ -3 & 6 \end{bmatrix} + X = \begin{bmatrix} 10 & -2 \\ 1 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. Find the values of *a* and *b* so that

$$\begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} a-b & 2 & -2 \\ 4 & a & b \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 2a+b & 5 \end{bmatrix}$$

8. For matrices A, B and C

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 5 & 7 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 3 & 7 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 6 \\ 7 & 1 \\ 4 & 1 \end{bmatrix}$$

   verify that A+(B+C) = (A+B)+C

9. If $A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$, find AB and BA. Is AB= BA?

10. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$, find AB and BA. Is AB = BA?

11. If $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$, find A^2.

12. Find A (B+C), if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} -2 & 0 & 3 \\ 4 & 0 & -3 \end{bmatrix}$$

13. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, find the values of x and y.

14. Show that $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the matrix equation $A^2 + 4A - 2I = O$.

Find inverse of the following matrices using elementary transformations.

15. $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

18. $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

20. $\begin{bmatrix} \cos x & \sin x \\ \sin x & \cos x \end{bmatrix}$

21. $\begin{bmatrix} 1 & \tan\frac{x}{2} \\ -\tan\frac{x}{2} & 1 \end{bmatrix}$

22. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 0 & 1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

24. $\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

ANSWERS

CHECK YOUR PROGRESS 20.1

1. $\begin{pmatrix} 56 & 65 & 71 \\ 29 & 37 & 57 \end{pmatrix}; \begin{pmatrix} 56 & 29 \\ 65 & 37 \\ 71 & 57 \end{pmatrix}$

2. $\begin{pmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{pmatrix}; \begin{pmatrix} 40 & 10 \\ 35 & 5 \\ 25 & 8 \end{pmatrix}$

3. $\begin{pmatrix} 4 & 6 & 3 \\ 4 & 3 & 5 \end{pmatrix}$

4. (a) 6 (b) 12 (c) 8 (d) 12 (e) ab (f) *mn*

5. (a) $1 \times 8; 2 \times 4; 4 \times 2; 8 \times 1$ (b) $1 \times 5; 5 \times 1$
   (c) $1 \times 12; 2 \times 6; 3 \times 4; 4 \times 3; 6 \times 2; 12 \times 1$
   (d) $1 \times 16; 2 \times 8; 4 \times 4; 8 \times 2; 16 \times 1$

6. (a) 4 (b) 5 (c) 4×5 (d) 20
   (e) $a_{14} = 0$; $a_{23} = 7$; $a_{34} = -3$; $a_{45} = 1$ and $a_{33} = 3$

7. (a) $\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 4 & 2 & \frac{4}{3} \\ 9 & \frac{9}{2} & 3 \end{pmatrix}$ (c) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \\ \frac{25}{2} & \frac{49}{2} & \frac{81}{2} \end{bmatrix}$ (d) $\begin{pmatrix} 1 & 4 & 7 \\ -1 & 2 & 5 \\ -3 & 0 & 3 \end{pmatrix}$

8. (a) $\begin{pmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 10 \\ 10 & 20 \\ 15 & 30 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$

CHECK YOUR PROGRESS 20.2

1. (a) G (b) B (c) A, D, E and F (d) A, D and F
   (e) D and F (f) F (g) C

2. (a) $a = 2$, $b = 10$, $c = 6$, $d = -2$
   (b) $a = 2$, $b = 3$, $c = 2$, $d = 5$
   (c) $a = \frac{3}{2}$, $b = -2$, $c = 2$, $d = -4$

3. No 4. No

CHECK YOUR PROGRESS 20.3

1. (a) $\begin{bmatrix} 28 & 8 \\ 8 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -7 & -2 \\ -2 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{7}{2} & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-21}{2} & -3 \\ -3 & \frac{-9}{2} \end{bmatrix}$

2. (a) $\begin{bmatrix} 0 & -5 & 10 \\ 15 & 5 & 20 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3 & -6 \\ -9 & -3 & -12 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & \frac{-1}{3} & \frac{2}{3} \\ 1 & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$

(d) $\begin{bmatrix} 0 & \frac{1}{2} & -1 \\ \frac{-3}{2} & \frac{-1}{2} & -2 \end{bmatrix}$ 3. $\begin{bmatrix} 7 & 0 \\ -28 & -14 \\ 0 & 7 \end{bmatrix}$

4. (a) $\begin{bmatrix} 15 & 0 & 5 \\ 20 & -10 & 0 \\ -5 & 0 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} -12 & 0 & -4 \\ -16 & 8 & 0 \\ 4 & 0 & -20 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ \frac{4}{3} & \frac{-2}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{5}{3} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-3}{2} & 0 & \frac{-1}{2} \\ -2 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-5}{2} \end{bmatrix}$

5. (a) $\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 10 & 8 \\ 9 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 & -6 \\ -2 & -1 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

CHECK YOUR PROGRESS 20.4

1. (a) $\begin{bmatrix} 3 & -2 \\ 8 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -3 \\ 13 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -4 \\ 14 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & -5 \\ 19 & 10 \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 3 & 6 \\ -5 & 3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -3 & -6 \\ 5 & -3 & -5 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 9 \\ -9 & 2 & 10 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & -11 & -15 \\ 11 & -10 & -10 \end{bmatrix}$

3. (a) $\begin{bmatrix} 0 & -6 & 3 \\ 5 & 5 & 3 \\ 6 & 5 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 & 3 \\ 3 & -7 & 1 \\ 2 & 5 & -7 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -2 & -3 \\ -3 & 7 & -1 \\ -2 & -5 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -14 & 9 \\ 14 & 9 & 8 \\ 16 & 15 & 14 \end{bmatrix}$

4. (a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. (a) $\begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & -3 \\ 4 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. (a) $\begin{bmatrix} 2 & 18 \\ 6 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 15 & 3 \\ 21 & 27 \end{bmatrix}$ (c) $\begin{bmatrix} 17 & 21 \\ 27 & 31 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-17}{5} & \frac{-21}{5} \\ \frac{-27}{5} & \frac{-31}{5} \end{bmatrix}$

7. (a) $\begin{bmatrix} 2 & 4 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ 0 & -4 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 6 & -1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$

We observe that $(A+B)' = B' + A'$

8. (a) $\begin{bmatrix} 0 & 2 \\ -5 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 \\ -5 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 8 \\ 3 & -7 \end{bmatrix}$

(e) $\begin{bmatrix} -2 & 3 \\ -8 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} -5 & 9 \\ -16 & -3 \end{bmatrix}$

CHECK YOUR PROGRESS 20.5

1. $AB = [-6]$; $BA = \begin{bmatrix} 0 & 0 & 0 \\ -4 & -6 & 0 \\ 2 & 3 & 0 \end{bmatrix}$ $AB \neq BA$

2. $AB = \begin{bmatrix} 2 & 2 \\ 1 & -6 \end{bmatrix}$; $BA = \begin{bmatrix} -3 & 13 & -4 \\ 1 & -1 & -2 \\ 2 & -6 & 0 \end{bmatrix}$ $AB \neq BA$

3. $AB = \begin{bmatrix} ax & ay & az \\ bx & by & bz \end{bmatrix}$; BA does not exist.

4. $BA = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$; AB does not exist.

5. Both AB and BA do not exist. AB does not exist since the number of columns of A is not equal to the number of rows of B. BA also does not exist since number of coluumns of B is not equal to the number of rows of A.

6. $AB = \begin{bmatrix} 0 & 5 \\ 6 & 15 \end{bmatrix}$; $BA = \begin{bmatrix} -2 & -1 \\ 4 & 17 \end{bmatrix}$; $AB \neq BA$

7. $AB = \begin{bmatrix} 4 & -3 & 7 \\ 3 & 17 & 24 \\ 14 & -13 & 17 \end{bmatrix}$; $BA = \begin{bmatrix} 16 & -8 & -11 \\ 16 & 11 & 3 \\ 10 & 21 & 11 \end{bmatrix}$; $AB \neq BA$.

8. $AB = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$; $BA = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$; $AB \neq BA$.

9. (a) $x = 3,\ y = -1$ (b) $x = -1,\ y = 2$

12. (a) $\begin{bmatrix} -14 & 18 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -14 & 18 \\ 2 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & 0 \\ 7 & 8 \end{bmatrix}$ (f) $\begin{bmatrix} -2 & -3 \\ 9 & 15 \end{bmatrix}$

13. (a) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$; AC= BC

Here, $A \neq B$ and $C \neq O$, yet AC= BC

i.e. cancellation law does not hold good for matrices.

14. (a) $\begin{pmatrix} 4 & 7 \\ 9 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & -7 \\ -14 & 9 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 1 \\ -3 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} -3 & -8 \\ -11 & 10 \end{pmatrix}$ (e) $\begin{pmatrix} -4 & -7 \\ -14 & 9 \end{pmatrix}$

We observe that A(B+C) = AB+AC

16. x= $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ 18. (a) No (b) No (c) No

CHECK YOUR PROGRESS 20.6

1. (*a*) $\frac{1}{25}\begin{bmatrix} 3 & 1 \\ 4 & -7 \end{bmatrix}$ (*b*) $\frac{1}{23}\begin{bmatrix} 5 & -6 \\ 3 & 1 \end{bmatrix}$ (*c*) does not exist

(*d*) $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$ (*e*) $\begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & 12 & 9 \end{bmatrix}$

TERMIAL EXERCISE

1. (a) 2 (b) 6 (c) 9 (d) 12

2. (a) $\begin{pmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ 5 & 4 \\ 8 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{5}{2} & 4 \\ \frac{7}{2} & 5 \\ \frac{9}{2} & 6 \end{pmatrix}$

3. (a) 3×1 (b) 1×3

(c) 3×2 (d) 2×3

4. (a) $x = 1, y = 2, z = 3$ (b) $x = 5, y = 1, z = 5$

(c) $x = 3, y = -3, z = 3$ (d) $x = 2, y = 1, z = 5$

5. (a) $\begin{bmatrix} 3 & 2 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -4 \\ 8 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -8 \\ 9 & 0 \end{bmatrix}$

6. (a) $\begin{bmatrix} 6 & -7 \\ 4 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 4 & -3 \\ -3 & 0 & -1 \end{bmatrix}$

7. $a = \dfrac{3}{2}$ $b = -\dfrac{3}{2}$

9. $AB = \begin{bmatrix} 13 & 11 \\ 38 & 43 \end{bmatrix}$; $BA = \begin{bmatrix} 5 & 10 & 17 \\ 6 & 14 & 24 \\ 4 & 21 & 37 \end{bmatrix}$; $AB \neq BA$

10. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $AB = BA$

11. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 9 & 1 & 1 \\ -1 & -4 & 10 \end{bmatrix}$

13. $x = 1, y = -4$.

15. $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

16. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$

18. $\dfrac{1}{22}\begin{bmatrix} -4 & +5 \\ +2 & +3 \end{bmatrix}$

19. $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

20. $\begin{bmatrix} \cos x & -\sin x \\ -\sin x & \cos x \end{bmatrix}$

21. $\cos^2 \dfrac{x}{2}\begin{bmatrix} 1 & -\tan {x}/{2} \\ \tan {x}/{2} & 1 \end{bmatrix}$

22. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

23. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix}$

21

DETERMINANTS

Every square matrix is associated with a unique number called the determinant of the matrix.

In this lesson, we will learn various properties of determinants and also evaluate determinants by different methods.

OBJECTIVES

After studying this lesson, you will be able to :

- define determinant of a square matrix;
- define the minor and the cofactor of an element of a matrix;
- find the minor and the cofactor of an element of a matrix;
- find the value of a given determinant of order not exceeding 3;
- state the properties of determinants;
- evaluate a given determinant of order not exceeding 3 by using expansion method;

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of solution of equations
- Knowledge of number system including complex number
- Four fundamental operations on numbers and expressions

21.1 DETERMINANT OF ORDER 2

Let us consider the following system of linear equations:

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

On solving this system of equations for x and y, we get

$$x = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1} \text{ and } y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} \text{ provided } a_1 b_2 - a_2 b_1 \neq 0$$

The number $a_1 b_2 - a_2 b_1$ determines whether the values of x and y exist or not.

The number $a_1b_2 - a_2b_1$ is called the value of the determinant, and we write

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

i.e. a_{11} belongs to the 1st row and 1st column
a_{12} belongs to the 1st row and 2nd column
a_{21} belongs to the 2nd row and 1st column
a_{22} belongs to the 2nd row and 2nd column

21.2 EXPANSION OF A DETERMINANT OF ORDER 2

A formal rule for the expansion of a determinant of order 2 may be stated as follows:

In the determinant, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

write the elements in the following manner :

$$\begin{matrix} a_1 & & a_{12} \\ & \times & \\ a_{21} & & a_{22} \end{matrix}$$

Multiply the elements by the arrow. The sign of the arrow going **downwards** is positive, i.e., a_{11} a_{22} and the sign of the arrow going **upwards** is negative, i.e., $-a_{21}$ a_{12}
Add these two products, i.e., a_{11} a_{22} $+(-a_{21}.a_{12})$ or a_{11} a_{22} $-a_{21}$ a_{12} which is the required value of the determinant.

Example 21.1 Evaluate :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix}$ (iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix}$

Solution :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix} = (6 \times 2) - (8 \times 4) = 12 - 32 = -20$

(ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix} = (a+b)\,(a+b) - (2a)\,(2b)$

$= a^2+2ab+b^2 - 4ab = a^2+b^2 - 2ab = (a-b)^2$

(iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix} = (x^2+x+1)\,(x-1) - (x^2-x+1)\,(x+1)$

$= (x^3-1) - (x^3+1) = -2$

Example 21.2 Find the value of x if

(i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ (ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

Solution :

(i) Now, $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$= (x^2 - 9) - x^2 - x = -x - 9$

According to the question,

$-x - 9 = 6$

$\Rightarrow \quad x = -15$

(ii) Now, $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$= 8x^2 + 4x - 4x - 2 - 2x^2 - x - 2x - 1$

$= 6x^2 - 3x - 3 = 3(2x^2 - x - 1)$

According to the equation

$3(2x^2 - x - 1) = 0$

or, $2x^2 - x - 1 = 0$

or, $2x^2 - 2x + x - 1 = 0$

or, $2x(x-1) + 1(x-1) = 0$

or, $(2x+1)(x-1) = 0$

or, $x = 1, -\frac{1}{2}$

21.3 DETERMINANT OF ORDER 3

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ contains nine quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 aranged in 3 rows and 3 columns, is called determinant of order 3 (or a determinant of third order). A determinant of order 3 has $(3)^2 = 9$ elements.

Using double subscript notations, viz., $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ for the elements

$a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3, we write a determinant of order3 as follows: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Usually a determinant, whether of order 2 or 3, is denoted by Δ or $|A|$, $|B|$ etc.

$$\Delta = |a_{ij}|, \text{ where } i = 1, 2, 3 \text{ and } j = 1, 2, 3$$

21.4 DETERMINANT OF A SQUARE MATRIX

With each square matrix of numbers (we associate) a "determinant of the matrix".
With the 1×1 matrix $[a]$, we associate the determinant of order 1 and with the only element a. The value of the determinant is a.

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a square matrix of order 2, then the expression $a_{11}a_{22} - a_{21}a_{12}$ is defined as the determinant of order 2. It is denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

With the 3×3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, we associate the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and its value is defined to be

$$a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)\, a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 21.3 If $A = \begin{pmatrix} 3 & 6 \\ 1 & 5 \end{pmatrix}$, find $|A|$

Solution : $|A| = \begin{vmatrix} 3 & 6 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 1 \times 6 = 15 - 6 = 9$

Example 21.4 If $A = \begin{pmatrix} a+b & a \\ b & a-b \end{pmatrix}$, find $|A|$

Solution : $|A| = \begin{vmatrix} a+b & a \\ b & a-b \end{vmatrix} = (a+b)(a-b) - b \times a = a^2 - b^2 - ab$

Note : 1. The determinant of a unit matrix I is 1.

2. A square matrix whose determinant is zero, is called the singular matrix.

21.5 EXPANSION OF A DETERMINANT OF ORDER 3

In Section 4.4, we have written

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which can be further expanded as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

We notice that in the above method of expansion, each element of first row is multiplied by the second order determinant obtained by deleting the row and column in which the element lies.

Further, mark that the elements a_{11}, a_{12} and a_{13} have been assigned positive, negative and positive signs, respectively. In other words, they are assigned positive and negative signs, alternatively, starting with positive sign. If the sum of the subscripts of the elements is an even number, we assign positive sign and if it is an odd number, then we assign negative sign. Therefore, a_{11} has been assigned positive sign.

Note : We can expand the determinant using any row or column. The value of the determinant will be the same whether we expand it using first row or first column or any row or column, taking into consideration rule of sign as explained above.

Example 21.5 Expand the determinant, using the first row

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$

$= 1 \text{ x } (20 - 2) - 2 \times (10 - 3) + 3 \times (4 - 12)$

$= 18 - 14 - 24$

$= -20$

Example 21.6 Expand the determinant, by using the second column

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-1) \times 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + (-1)\, 3 \times \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}$

$= -2 \times (3 - 4) + 1 \times (1 - 6) - 3 \times (2 - 9)$

$= 2 - 5 + 21$

$= 18$

CHECK YOUR PROGRESS 21.1

1. Find $|A|$, if

(a) $A = \begin{bmatrix} 2+\sqrt{3} & 5 \\ 2 & 2-\sqrt{3} \end{bmatrix}$ (b) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

(c) $A = \begin{bmatrix} \sin\alpha + \cos\beta & \cos\beta + \cos\alpha \\ \cos\beta - \cos\alpha & \sin\alpha - \sin\beta \end{bmatrix}$ (d) $\begin{bmatrix} a+bi & c+di \\ c-di & a-bi \end{bmatrix}$

2. Find which of the following matrices are singular matrices :

(a) $\begin{bmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$

3. Expand the determinant by using first row

(a) $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$ (d) $\begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}$

21.6 MINORS AND COFACTORS

21.6.1 *Minor of a_{ij} in |A|*

To each element of a determinant, a number called its minor is associated.

The minor of an element is the value of the determinant obtained by deleting the row and column containing the element.

Thus, the minor of an element a_{ij} in |A| is the value of the determinant obtained by deleting the i^{th} row and j^{th} column of |A| and is denoted by M_{ij}. For example, minor of 3 in the determinant

$\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$ is 7.

Example 21.7 Find the minors of the elements of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

Let M_{ij} denote the minor of a_{ij}. Now, a_{11} occurs in the 1^{st} row and 1^{st} column. Thus to find the minor of a_{11}, we delete the 1^{st} row and 1^{st} column of |A|.

The minor M_{11} of a_{11} is given by

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Similarly, the minor M_{12} of a_{12} is given by

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} \ ; \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13} \ ; \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

Similarly we can find M_{31}, M_{32} and M_{33}.

21.6.2 *Cofactors of a_{ij} in $|A|$*

The cofactor of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$. It is usually denoted by C_{ij}. Thus,

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

Example 21.8 Find the cofactors of the elements a_{11}, a_{12}, and a_{21} of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

The cofactor of any element a_{ij} is $(-1)^{i+j} M_{ij}$, then

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (a_{22}\, a_{33} - a_{32}\, a_{23})$$
$$= (a_{22}\, a_{33} - a_{32}\, a_{23})$$
$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_{21}\, a_{33} - a_{31}\, a_{23}) = (a_{31}a_{23} - a_{21}a_{33})$$

and $C_{21} = (-1)^{2+1} M_{21} = -M_{21} = (a_{32}\, a_{13} - a_{12}\, a_{33})$

Example 21.9 Find the minors and cofactors of the elements of the second row in the determinant

$$|A| = \begin{vmatrix} 1 & 6 & 3 \\ 5 & 2 & 4 \\ 7 & 0 & 8 \end{vmatrix}$$

Solution : The elements of the second row are $a_{21}=5$; $a_{22}=2$; $a_{23}=4$.

$$\text{Minor of } a_{21} \text{ (i.e., 5)} = \begin{vmatrix} 6 & 3 \\ 0 & 8 \end{vmatrix} = 48 - 0 = 48$$

$$\text{Minor of } a_{22} \text{ (i.e., 2)} = \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = 8 - 21 = -13$$

$$\text{and Minor of } a_{23} \text{ (i.e., 4)} = \begin{vmatrix} 1 & 6 \\ 7 & 0 \end{vmatrix} = 0 - 42 = -42$$

The corresponding cofactors will be

$$C_{21} = (-1)^{2+1} M_{21} = -(48) = -48$$

$$C_{22} = (-1)^{2+2} M_{22} = +(-13) = -13$$

and $$C_{23} = (-1)^{2+3} M_{23} = -(-42) = 42$$

CHECK YOUR PROGRESS 21.2

1. Find the minors and cofactors of the elements of the second row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

2. Find the minors and cofactors of the elements of the third column of the determinat

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

3. Evaluate each of the following determinants using cofactors:

(a) $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & -4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (e) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ (f) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

4. Solve for x, the following equations:

(a) $\begin{vmatrix} x & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x & 3 & 3 \\ 3 & 3 & x \\ 2 & 3 & 3 \end{vmatrix} = 0$ (c) $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$

21.7 PROPERTIES OF DETERMINANTS

We shall now discuss some of the properties of determinants. These properties will help us in expanding the determinants.

Property 1: The value of a determinant remains unchanged if its rows and columns are interchanged.

$$\text{Let } \Delta = \begin{vmatrix} 2 & -1 & 3 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first column, we have

$$\Delta \quad = 2\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} - 0\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ -3 & 0 \end{vmatrix}$$

$$= 2\,(3-0) - 0\,(1-6) + 4\,(0+9)$$

$$= 6 + 36 = 42$$

Let Δ' be the determinant obtained by interchanging rows and columns of Δ. Then

$$\Delta' = \begin{vmatrix} 2 & 0 & 4 \\ -1 & -3 & 2 \\ 3 & 0 & -1 \end{vmatrix}$$

Expanding the determinant Δ' by second column, we have (Recall that a determinant can be expanded by any of its rows or columns)

$$(-)\,0\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + (-)\,0\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= 0 + (-3)\,(-2-12) + 0$$

$$= 42$$

Thus, we see that $\Delta = \Delta'$

Property 2: If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$= 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 2\,(4-3) - 3\,(2-9) + 1\,(1-6)$$

$$= 2 + 21 - 5 = 18$$

Let Δ' be the determinant obtained by interchanging C_1 and C_2

Then $\Delta' = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$

Expanding the determinant Δ' by first row, we have

$$3\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3\,(2-9) - 2(4-3) + 1(6-1)$$

$$= -21-2+5 = -18$$

Thus we see that $\Delta' = -\Delta$

Corollary

If any row (or a column) of a determinant is passed over 'n' rows (or columns), then the resulting determinant Δ' is $\Delta = (-1)^n\ \Delta$

For example, $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 5 & 6 \\ 0 & 4 & 2 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 5 & 6 \\ 0 & 4 & 2 \\ 2 & 3 & 5 \end{vmatrix}$

$$= 2\,(10-24) - 3\,(2-0) + 5(4)$$

$$= -28 - 6 + 20 = -14$$

Property 3: If any two rows (or columns) of a determinant are identical then the value of the determinant is zero.

Proof : Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

be a determinant with identical columns C_1 and C_2 and let Δ' determinant obtained from Δ by interchanging C_1 and C_2

Then,

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is the same as Δ, but by property 2, the value of the determinant changes in sign, if its any two adjacent rows (or columns) are interchnaged

Therefore $\quad \Delta' = -\Delta$

Thus, we find that

or $\quad 2\Delta = 0 \Rightarrow \Delta = 0$

Hence the value of a determinant is zero, if it has two identical rows (or columns).

Property 4: If each element of a row (or column) of a determinant is multiplied by the same constant say, $k \neq 0$, then the value of the determinat is multiplied by that constant k.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$\Delta = 2(3-0) - 1\,(0-0) + (-5)\,(0+12\,)$$

$$= 6 - 60 = -54$$

Let us multiply column 3 of Δ by 4. Then, the new determinant Δ' is :

$$\Delta' = \begin{vmatrix} 2 & 1 & -20 \\ 0 & -3 & 0 \\ 4 & 2 & -4 \end{vmatrix}$$

Expanding the determinant Δ' by first row, we have

$$\Delta' = 2\,(12-0) - 1\,(0-0) + (-20)\,(0+12)$$

$$= 24 - 240 = -216$$

$$= 4\,\Delta$$

Corollary :
If any two rows (or columns) of a determinant are proportional, then its value is zero.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & ka_1 \\ a_2 & b_2 & ka_2 \\ a_3 & b_3 & ka_3 \end{vmatrix}$

Note that elements of column 3 are k times the corresponding elements of column 1

By Property 4, $\Delta = k \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$

$= k \times 0$ (by Property 2)

$= 0$

Property 5: If each element of a row (or of a column) of a determinant is expressed as the sum (or difference) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order whose remaining rows (or columns) do not change.

Proof : Let $\Delta = \begin{vmatrix} a_1+\alpha & b_1+\beta & c_1+\gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then, on expanding the determinant by the first row, we have

$$\Delta = (a_1+\alpha)(b_2c_3-b_3c_2)-(b_1+\beta)(a_2c_3-a_3c_2)+(c_1+\gamma)(a_2b_3-a_3b_2)$$

$$= a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)+\alpha(b_2c_3-b_3c_2)$$

$$-\beta(a_2c_3-a_3c_2)+\gamma(a_2b_3-a_3b_2)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Thus, the determinant Δ can be expressed as the sum of the determinants of the same order.

Property 6: The value of a determinant does not change, if to each element of a row (or a column) be added (or subtracted) the some multiples of the corresponding elements of one or more other rows (or columns)

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Δ' be the determinant obtained from Δ by corresponding elements of R_3

i.e. $R_1 \to R_1 + kR_3$

Then, $$\Delta' = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

or, $\Delta' = \Delta + k \times 0$ (Row 1 and Row 3 are identical)

$\Delta' = \Delta$

21.8 EVALUATION OF A DETERMINANT USING PROPERTIES

Now we are in a position to evaluate a determinant easily by applying the aforesaid properties. The purpose of simplification of a determinant is to make maximum possible zeroes in a row (or column) by using the above properties and then to expand the determinant by that row (or column). We denote 1st, 2nd and 3rd row by R_1, R_2, and R_3 respectively and 1st,2nd and 3rd column by C_1, C_2 and C_3 respectively.

Example 21.10 Show that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

where w is a non-real cube root of unity.

Solution : $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

Add the sum of the 2nd and 3rd column to the 1st column. We write this operation as $C_1 \to C_1 + (C_2 + C_3)$

$$\therefore \qquad \Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} = 0 \quad \text{(on expanding by } C_1\text{)}$$

(since w is a non-real cube root of unity, therefore, $1+w+w^2=0$)

Example 21.11 Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)\,(b-c)\,(c-a)$

Solution : $\qquad \Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \begin{vmatrix} 0 & a-c & bc-ab \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix} [R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$$

$$= \begin{vmatrix} 0 & a-c & b(c-a) \\ 0 & b-c & a(c-b) \\ 1 & c & ab \end{vmatrix} = (a-c)(b-c)\begin{vmatrix} 0 & 1 & -b \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$$

Expanding by C_1, we have

$$\Delta \qquad = (a-c)(b-c)\begin{vmatrix} 1 & -b \\ 1 & -a \end{vmatrix} = (a-c)(b-c)(b-a)$$

$$= (a-b)(b-c)(c-a)$$

Example 21.12

Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$

Solution : $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad R_1 \to R_1 - (R_2 + R_3)$$

Expanding by R_1, we get

$$= 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} - 2b\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2c\,[b(a+b) - bc] - 2b[bc - c(c+a)]$$

$$= 2bc\,[a+b-c] - 2bc[b-c-a]$$

$$= 2bc\,[(a+b-c) - (b-c-a)]$$

$$= 2bc\,[a+b-c-b+c+a]$$

$$= 4abc$$

Example 21.13 Evaluate:

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3 = 0,$$

Example 21.14 Prove that

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution : Δ

$$= \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bc & bc+ab+ac \\ 1 & ca & ca+bc+ba \\ 1 & ab & ab+ca+cb \end{vmatrix} \qquad C_3 \to C_2 + C_3$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$$= (ab+bc+ca) \times 0 \qquad \text{(by Property 3)}$$

$$= 0$$

Example 21.15 Show that

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : Δ

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & b & c \\ 0 & 0 & 2c \\ 0 & 2b & 0 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$= abc(-a)\begin{vmatrix} 0 & 2c \\ 2b & 0 \end{vmatrix} \quad \text{(on expanding by } C_1\text{)}$$

$$= abc(-a)(-4bc)$$

$$= 4a^2b^2c^2$$

Example 21.16 Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2\,(a+3)$$

Solution : Δ

$$= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= \begin{vmatrix} a+3 & 1 & 1 \\ a+3 & 1+a & 1 \\ a+3 & 1 & 1+a \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3$$

$$= (a+3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (a+3)\begin{vmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 1 & 0 & a \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (a+3) \times (1)\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix}$$

$$= (a+3)(a^2)$$

$$= a^2(a+3)$$

CHECK YOUR PROGRESS 21.3

1. Show that $\begin{vmatrix} x+3 & x & x \\ x & x+3 & x \\ x & x & x+3 \end{vmatrix} = 27\,(x+1)$

2. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

3. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = bc+ca+ab+abc$

4. Show that $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$

5. Show that $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

6. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

7. Evaluate

(a) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ (b) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

8. Solve for x :

$$\begin{vmatrix} 3x-8 & 3 & x \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

21.11 Application of Determinants

Determinant is used to find area of a triangle.

21.11.1 Area of a Triangle

We know that area of a triangle ABC, (say) whose vertices are $(x_1 y_1)$, $(x_2 y_2)$ and $(x_3 y_3)$ is given by

$$\text{Area of } (\Delta ABC) = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right] \quad ...(i)$$

Also, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$ [expanding along C_1]

$$= x_1(y_2-y_3)-x_2(y_1-y_3)+x_3(y_1-y_2)$$

$$= x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2) \quad ...(ii)$$

from (*i*) and (*ii*)

$$\text{Area } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & x_2 & 1 \\ y_3 & y_3 & 1 \end{vmatrix}$$

Thus the area of a triangle having vertices as $(x_1\ y_1)$, $(x_2\ y_2)$ and $(x_3\ y_3)$ is given by

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & x_2 & 1 \\ y_3 & y_3 & 1 \end{vmatrix}$$

21.11.2 Condition of collinearity of three points :

Let A(x_1 y_1), B(x_2 y_2) and C(x_3 y_3) be three points then

A, B, C are collinear if area of ΔABC = 0

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

21.11.2 Equation of a line passing through the given two points

Let the two points be P(x_1, y_1) and Q(x_2 y_2) and R(x y) be any point on the line joining P and Q since the points P, Q and R are collinear.

$$\therefore \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Thus the equation of the line joining points $(x_1\, y_1)$ and $(x_1\, y_2)$ is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Example 21.17 Find the area of the triangle with vertices P(5, 4), Q(–2, 4) and R(2, –6)

Solution : Let A be the area of the triangle PQR, then

$$A = \frac{1}{2}\begin{vmatrix} 5 & 4 & 1 \\ -2 & 4 & 1 \\ 2 & -6 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[5(4-(-6)) - 4(-2-2) + 1(12-8)]$$

$$= \frac{1}{2}[50+16+4] = \frac{1}{2}(70) = 35 \text{ sq units.}$$

Example 21.18 Show that points (a, b + c), (b, c + a) and (c, a + b) are collinear.

Solution : We have

$$\Delta = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$c_2 \to c_2 + c_1$

$$= \begin{vmatrix} a & a+b+c & 1 \\ b & b+c+a & 1 \\ c & c+a+b & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = (a + b + c) \times 0 = 0$$

Hence, the given points are collinear.

Example 21.19 Find equation of the line joining A(1, 3) and B(2, 1) using determinants.

Solution : Let P(x, y) be any point on the line joining A(1, 3) and B(2, 1). Then

$$\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad x\,(3 - 1) - y\,(1 - 2) + 1\,(1 - 6) = 0$

$\Rightarrow \quad 2x + y - 5 = 0$

This is the required equation of line AB.

CHECK YOUR PROGRESS 21.4

1. Find area of the ΔABC when A, B and C are (3, 8), (–4, 2) and (5, –1) respectively.
2. Show that points A(5, 5), B(–5, 1) and C(10, 7) are collinear.
3. Using determinants find the equation of the line joining (1, 2) and (3, 6).

LET US SUM UP

- The expression $a_1b_2 - a_2b_1$ is denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
- With each square matrix, a determinant of the matrix can be associated.
- The *minor* of any element in a determinant is obtained from the given determinant by deleting the row and column in which the element lies.

- The *cofactor* of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$
- A determinant can be expanded using any row or column. The value of the determinant will be the same.
- A square matrix whose determinant has the value zero, is called a *singular matrix*.
- The value of a determinant remains unchanged, if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.
- If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- If each element of a row (or column) of a determinant is multiplied by the same constant, then the value of the determinant is multiplied by the constant.
- If any two rows (or columns) of a determinant are proportional, then its value is zero.
- If each element of a row or column from of a determinant is expressed as the sum (or differenence) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order.
- The value of a determinant does not change if to each element of a row (or column) be added to (or subtracted from) some multiples of the corresponding elements of one or more rows (or columns).
- Product of a matrix and its inverse is equal to identity matrix of same order.
- Inverse of a matrix is always unique.
- All matrices are not necessarily invertible.
- Three points are collinear if the area of the triangle formed by these three points is zero.

SUPPORTIVE WEB SITES

http://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=det

http://mathworld.wolfram.com/Determinant.html

http://en.wikipedia.org/wiki/Determinant

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Matrices_and_determinants.html

TERMINAL EXERCISE

1. Find all the minors and cofactors of $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ by expanding it using the first column.

3. Evaluate $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix}$ 4. Solve for x, if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$

5. Using properties of determinants, show that

(a) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

(b) $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

6. Evaluate: (a) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & w^3 & w^5 \\ w^3 & 1 & w^4 \\ w^5 & w^5 & 1 \end{vmatrix}$

,w being an imaginary cube-root of unity

7. Find the area of the triangle with vertices at the points :

(i) (2, 7), (1, 1) and (10, 8)   (ii) $(-1,-8),(-2,-3)$ and $(3,2)$

(ii) (0, 0) (6, 0) and (4, 3)   (iv) (1, 4), (2, 3) and (–5, –3)

8. Using determinants find the value of k so that the following points become collinear

(i) $(k, 2-2k), (-k+1, 2k)$ and $(-4-k, 6-2k)$

(ii) $(k,-2), (5,2)$ and $(6,8)$

(iii) $(3,-2), (k,2)$ and $(8,8)$

(iv) $(1,-5)(-4,5)(k,7)$

9. Using determinants, find the equation of the line joining the points

(i) (1, 2) and (3, 6)   (ii) (3, 1) and (9, 3)

10. If the points (a, 0), (0, b) and (1, 1) are collinear then using determinants show that $ab = a + b$

ANSWERS

CHECK YOUR PROGRESS 21.1

1. (a) 11 (b) 1 (c) 0 (d) $(a^2+b^2)-(c^2+d^2)$

2. (a) and (d)

3. (a) 18 (b) −54

(c) $adf + 2bce - ae^2 - fb^2 - de^2$ (d) $x-1$

CHECK YOUR PROGRESS 21.2

1. $M_{21} = 39;\ C_{21} = -39$

$M_{22} = 3;\ C_{22} = 3$

$M_{23} = -11;\ C_{23} = 11$

2. $M_{13} = -5;\ C_{13} = -5$

$M_{23} = -7;\ C_{23} = 7$

$M_{33} = 1;\ C_{33} = 1$

3. (a) 19 (b) 0 (c) –131

(d) $(a-b)(b-c)(c-a)$ (e) $4abc$ (f) 0

4. (a) $x = 2$ (b) $x = 2, 3$ (c) $x = 2, -\dfrac{17}{7}$

CHECK YOUR PROGRESS 21.3

7. (a) a^3 (b) $2abc(a+b+c)^3$

8. $x = \dfrac{2}{3}, \dfrac{11}{3}, \dfrac{11}{3}$

CHECK YOUR PROGRESS 21.4

1. $\dfrac{75}{2}$ sq units (3) $y = 2x$

TERMINAL EXERCISE

1. $M_{11} = -2,\ M_{12} = -1,\ M_{13} = 1,\ M_{21} = -7,\ M_{22} = -5,\ M_{23} = -1,$
$M_{31} = -8,\ M_{32} = -7,\ M_{33} = -2$

$C_{11} = -2,\ C_{12} = 1,\ C_{13} = 1,\ C_{21} = 7,\ C_{22} = -5,\ C_{23} = 1,$
$C_{31} = -8,\ C_{32} = 7,\ C_{33} = -2$

2. 0

3. –31

4. $x = 0,\ x = 1$

6. ( ) -8 (b) 0

7. (*i*) $\frac{45}{2}$ sq units (*ii*) 5 sq units

(*iii*) 9 sq units (*iv*) $\frac{15}{2}$ sq units

8. (*i*) $k = -1, \frac{1}{2}$ (*ii*) k = $\frac{13}{3}$

(*iii*) k = 5 (*iv*) k = –5

9. (*i*) $y = 2x$ (ii) $x = 3y$

22

INVERSE OF A MATRIX AND ITS APPLICATIONS

Let us Consider an Example:

Abhinav spends Rs. 120 in buying 2 pens and 5 note books whereas Shantanu spends Rs. 100 in buying 4 pens and 3 note books. We will try to find the cost of one pen and the cost of one note book using matrices.

Let the cost of 1 pen be Rs. x and the cost of 1 note book be Rs. y. Then the above information can be written in matrix form as:

$$\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 120 \\ 100 \end{bmatrix}$$

This can be written as $AX = B$

where $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 120 \\ 100 \end{bmatrix}$

Our aim is to find $X = \begin{bmatrix} x \\ y \end{bmatrix}$

In order to find X, we need to find a matrix A^{-1} so that $X = A^{-1}B$

This matrix A^{-1} is called the inverse of the matrix A.

In this lesson, we will try to find the existence of such matrices. We will also learn to solve a system of linear equations using matrix method.

OBJECTIVES

After studying this lesson, you will be able to :

- define a minor and a cofactor of an element of a matrix;
- find minor and cofactor of an element of a matrix;

- find the adjoint of a matrix;
- define and identify singular and non-singular matrices;
- find the inverse of a matrix, if it exists;
- represent system of linear equations in the matrix form $AX = B$; and
- solve a system of linear equations by matrix method.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a determinant.
- Determinant of a matrix.
- Matrix with its determinant of value 0.
- Transpose of a matrix.
- Minors and Cofactors of an element of a matrix.

22.1 DETERMINANT OF A SQUARE MATRIX

We have already learnt that with each square matrix, a determinant is associated. For any given matrix, say $A = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$

its determinant will be $\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix}$. It is denoted by $|A|$.

Similarly, for the matrix $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{pmatrix}$, the corresponding determinant is

$$|A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{vmatrix}$$

A square matrix A is said to be singular if its determinant is zero, i.e. $|A| = 0$

A square matrix A is said to be non-singular if its determinant is non-zero, i.e. $|A| \neq 0$

Example 22.1 Determine whether matrix A is singular or non-singular where

(a) $A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

Solution: (a) Here, $|A| = \begin{vmatrix} -6 & -3 \\ 4 & 2 \end{vmatrix}$

$$= (-6)(2) - (4)(-3)$$
$$= -12 + 12 = 0$$

Therefore, the given matrix A is a singular matrix.

(b)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= -7 + 4 - 3 = -6 \neq 0$$

Therefore, the given matrix is non-singular.

Example 22.2 Find the value of x for which the following matrix is singular:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$$

Solution: Here,

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix}$$

$$= 1(-6-2) + 2(-3-x) + 3(2-2x)$$
$$= -8 - 6 - 2x + 6 - 6x$$
$$= -8 - 8x$$

Since the matrix A is singular, we have $|A| = 0$

$$|A| = -8 - 8x = 0$$

$$or \quad x = -1$$

Thus, the required value of x is -1.

Example 22.3 Given $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix}$. Show that $|A| = |A'|$, where A' denotes the transpose of the matrix.

Solution: Here, $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix}$

This gives $A' = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}$

Now, $$|A| = \begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \qquad \text{...(1)}$$

and $$|A'| = \begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \qquad \text{...(2)}$$

From (1) and (2), we find that $|A| = |A'|$

22.2 MINORS AND COFACTORS OF THE ELEMENTS OF SQUARE MATRIX

Consider a matrix $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of the matrix obtained by deleting the ith row and jth column of A, is called the minor of a_{ij} and is denotes by M_{ij}.

Cofactor C_{ij} of a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, M_{23} = Minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

and C_{23} = Cofactor of a_{23}

$$= (-1)^{2+3} M_{23} = (-1)^5 M_{23} = -M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 22.4 Find the minors and the cofactors of the elements of matrix $A = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}$

Solution: For matrix A, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} = 6 - 30 = -24$

M_{11} (minor of 2) = 3; $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = 3$

M_{12} (minor of 5) = 6; $C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -6$

M_{21} (minor of 6) = 5; $C_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21} = -5$

M_{22} (minor of 3) = 2; $C_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22} = 2$

Example 22.5 Find the minors and the cofactors of the elements of matrix

$$A = \begin{bmatrix} -1 & 3 & 6 \\ 2 & 5 & -2 \\ 4 & 1 & 3 \end{bmatrix}$$

Solution: Here, $M_{11} = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15+2 = 17; \; C_{11} = (-1)^{1+1} M_{11} = 17$

$$M_{12} = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6+8 = 14; \; C_{12} = (-1)^{1+2} M_{12} = -14$$

$$M_{13} = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = 2-20 = -18; \; C_{13} = (-1)^{1+3} M_{13} = -18$$

$$M_{21} = \begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} = 9-6 = 3; \; C_{21} = (-1)^{2+1} M_{21} = -3$$

$$M_{22} = \begin{vmatrix} -1 & 6 \\ 4 & 3 \end{vmatrix} = (-3-24) = -27; \; C_{22} = (-1)^{2+2} M_{22} = -27$$

$$M_{23} = \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} = (-1-12) = -13; \; C_{23} = (-1)^{2+3} M_{23} = 13$$

$$M_{31} = \begin{vmatrix} 3 & 6 \\ 5 & -2 \end{vmatrix} = (-6-30) = -36; \; C_{31} = (-1)^{3+1} M_{31} = -36$$

$$M_{32} = \begin{vmatrix} -1 & 6 \\ 2 & -2 \end{vmatrix} = (2-12) = -10; \; C_{32} = (-1)^{3+2} M_{32} = 10$$

and $$M_{33} = \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = (-5-6) = -11; \; C_{33} = (-1)^{3+3} M_{33} = -11$$

CHECK YOUR PROGRESS 22.1

1. Find the value of the determinant of following matrices:

(a) $A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \end{bmatrix}$ (b) $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

2. Determine whether the following matrix are singular or non-singular.

(a) $A = \begin{bmatrix} 3 & 2 \\ -9 & -6 \end{bmatrix}$ (b) $Q = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & -1 \end{bmatrix}$

3. Find the minors of the following matrices:

(a) $A = \begin{bmatrix} 3 & -1 \\ 7 & 4 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 6 \\ 2 & 5 \end{bmatrix}$

4. (a) Find the minors of the elements of the 2nd row of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ -2 & -3 & 1 \end{bmatrix}$$

(b) Find the minors of the elements of the 3rd row of matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 1 \\ -2 & 0 & -3 \end{bmatrix}$$

5. Find the cofactors of the elements of each the following matrices:

(a) $A = \begin{bmatrix} 3 & -2 \\ 9 & 7 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 4 \\ -5 & 6 \end{bmatrix}$

6. (a) Find the cofactors of elements of the 2nd row of matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \\ 4 & 1 & -2 \end{bmatrix}$$

(b) Find the cofactors of the elements of the 1st row of matrix

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & 4 & -2 \\ -5 & -3 & 0 \end{bmatrix}$$

7. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 7 & 4 \end{bmatrix}$, verify that

(a) $|A| = |A'|$ and $|B| = |B'|$ (b) $|AB| = |A|\,|B| = |BA|$

22.3 ADJOINT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$ be a matrix. Then $|A| = \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix}$

Let M_{ij} and C_{ij} be the minor and cofactor of a_{ij} respectively. Then

$$M_{11} = |7| = 7;\ C_{11} = (-1)^{1+1}\,|7| = 7$$

$$M_{12} = |5| = 5;\ C_{12} = (-1)^{1+2}\,|5| = -5$$

$$M_{21} = |1| = 1;\ C_{21} = (-1)^{2+1}\,|1| = -1$$

$$M_{22} = |2| = 2;\ C_{22} = (-1)^{2+2}\ |2| = 2$$

We replace each element of A by its cofactor and get

$$B = \begin{bmatrix} 7 & -5 \\ -1 & 2 \end{bmatrix} \qquad \ldots(1)$$

The transpose of the matrix B of cofactors obtained in (1) above is

$$B' = \begin{bmatrix} 7 & -1 \\ -5 & 2 \end{bmatrix} \qquad \ldots(2)$$

The matrix B' obtained above is called the adjoint of matrix A. It is denoted by Adj A.

Thus, adjoint of a given matrix is the transpose of the matrix whose elements are the cofactors of the elements of the given matrix.

Working Rule: To find the Adj A of a matrix A:

(a) replace each element of A by its cofactor and obtain the matrix of cofactors; and

(b) take the transpose of the marix of cofactors, obtained in (a).

Example 22.6 Find the adjoint of

$$A = \begin{bmatrix} -4 & 5 \\ 2 & -3 \end{bmatrix}$$

Solution: Here, $|A| = \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix}$ Let A_{ij} be the cofactor of the element a_{ij}.

Then,

$$A_{11} = (-1)^{1+1}\ (-3) = -3 \qquad A_{21} = (-1)^{2+1}\ (5) = -5$$

$$A_{12} = (-1)^{1+2}\ (2) = -2 \qquad A_{22} = (-1)^{2+2}\ (-4) = -4$$

We replace each element of A by its cofactor to obtain its matrix of cofators as

$$\begin{bmatrix} -3 & -2 \\ -5 & -4 \end{bmatrix} \qquad \ldots(1)$$

Transpose of matrix in (1) is Adj A.

Thus, Adj $A = \begin{pmatrix} -3 & -5 \\ -2 & -4 \end{pmatrix}$

Example 22.7 Find the adjoint of $A = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{pmatrix}$

Solution: Here,

$$A = \begin{vmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{vmatrix}$$

Let A_{ij} be the cofactor of the element a_{ij} of $|A|$

Then $A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} = (-4-2) = -6$; $A_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ 5 & -1 \end{vmatrix} = -(3-5) = 2$

$A_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 4 \\ 5 & 2 \end{vmatrix} = (-6-20) = -26$; $A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = (-1-4) = 3$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} = (-1-10) = -11$; $A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} = -(2+5) = -7$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = (-1-8) = -9$; $A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = -(1+6) = -7$

and $A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} = (4-3) = 1$

Replacing the elements of A by their cofactors, we get the matrix of cofactors as

$\begin{pmatrix} -6 & 2 & -26 \\ 3 & -11 & -7 \\ -9 & -7 & 1 \end{pmatrix}$ Thus, Adj $A = \begin{pmatrix} -6 & 3 & -9 \\ 2 & -11 & -7 \\ -26 & -7 & 1 \end{pmatrix}$

If A is any square matrix of order n, then A(Adj A) = (Adj A) A = |A| I_n

where I_n is the unit matrix of order n.

MODULE - VI
Algebra-II

Notes

Verification:

(1) Consider $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix}$ or $|A| = 2 \times 3 - (-1) \times (4) = 10$

Here, $A_{11}=3;\ A_{12}=1;\ A_{21}=-4$ and $A_{22}=2$

Therefore, $\text{Adj } A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$

Now, $A\,(\text{Adj}A) = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|\,I_2$

(2) Consider, $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Then, $|A| = 3(-6-1) - 5\,(4-1) + 7\,(2+3) = -1$

Here, $A_{11} = -7;\ A_{12} = -3;\ A_{13}=5$

$A_{21} = -3;\ A_{22} = -1;\ A_{23}=2$

$A_{31}=26;\ A_{32}=11;\ A_{33} = -19$

Therefore, $\text{Adj } A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

Now $(A)\,(\text{Adj } A) = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3$$

Also, $\quad (\text{Adj A})\,A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix} \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$$

Note : If A is a singular matrix, i.e. $|A|=0$, then $A(\text{Adj A}) = O$

CHECK YOUR PROGRESS 22.2

1. Find adjoint of the following matrices:

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

2. Find adjoint of the following matrices :

(a) $\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & -i \\ i & i \end{bmatrix}$

Also verify in each case that $A(\text{Adj A}) = (\text{Adj A})A = |A| I_2$.

3. Verify that

$A(\text{Adj A}) = (\text{Adj A})A = |A|\, I_3$, where A is given by

(a) $\begin{bmatrix} 6 & 8 & -1 \\ 0 & 5 & 4 \\ -3 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 7 & 9 \\ 0 & -1 & 2 \\ 3 & -7 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}$

22.4 INVERSE OF A MATRIX

Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We will find, if possible, a matrix

$$B = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \text{ such that } AB = BA = I$$

i.e.,
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or
$$\begin{pmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

On comparing both sides, we get

$$ax + bu = 1 \quad ay + bv = 0$$

$$cx + du = 0 \qquad cy + dv = 1$$

Solving for x, y, u and v, we get

$$x = \frac{d}{ad - bc}, \; y = \frac{-b}{ad - bc}, \; u = \frac{-c}{ad - bc}, \; v = \frac{a}{ad - bc}$$

provided $ad - bc \neq 0$, i.e. , $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

Thus,
$$B = \begin{pmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{pmatrix}$$

or
$$B = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

It may be verified that $BA = I$.

It may be noted from above that, we have been able to find a matrix.

$$B = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{|A|} Adj\ A \qquad ...(1)$$

This matrix B, is called the inverse of A and is denoted by A^{-1}.

For a given matrix A, if there exists a matrix B such that $AB = BA = I$, then B is called the multiplicative inverse of A. We write this as $B = A^{-1}$.

Note: Observe that if $ad - bc = 0$, i.e., $|A| = 0$, the R.H.S. of (1) does not exist and $B\ (=A^{-1})$ is not defined. This is the reason why we need the matrix A to be non-singular in order that A possesses multiplicative inverse. Hence only non-singular matrices possess multiplicative inverse. Also B is non-singular and $A=B^{-1}$.

Example 22.8 Find the inverse of the matrix

$$A = \begin{pmatrix} 4 & 5 \\ 2 & -3 \end{pmatrix}$$

Solution : $$A = \begin{pmatrix} 4 & 5 \\ 2 & -3 \end{pmatrix}$$

Therefore, $|A| = -12 - 10 = -22 \neq 0$

$\therefore A$ is non-singular. It means A has an inverse. i.e. A^{-1} exists.

Now, $$\text{Adj } A = \begin{pmatrix} -3 & -5 \\ -2 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-22}\begin{bmatrix} -3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{22} & \frac{5}{22} \\ \frac{1}{11} & -\frac{2}{11} \end{bmatrix}$$

Note : Verify that $AA^{-1} = A^{-1}A = I$

Example 22.9 Find the inverse of matrix

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{pmatrix}$$

Solution : Here, $$A = \begin{pmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{pmatrix}$$

$$\begin{aligned} \therefore \quad |A| &= 3(5 - 24) - 2(-5 - 30) - 2(4 + 5) \\ &= 3(-19) - 2(-35) - 2(9) \\ &= -57 + 70 - 18 \\ &= -5 \neq 0 \end{aligned}$$

$\therefore \quad A^{-1}$ exists.

Let A_{ij} be the cofactor of the element a_{ij}.

Then,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 6 \\ 4 & -5 \end{vmatrix} = 5-24 = -19,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 6 \\ 5 & -5 \end{vmatrix} = -(-5-30) = 35.$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 5 & 4 \end{vmatrix} = 4-5 = 9,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -2 \\ 4 & -5 \end{vmatrix} = -(-10+8) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -2 \\ 5 & -5 \end{vmatrix} = -15+10 = -5,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = -(12-10) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -2 \\ -1 & 6 \end{vmatrix} = 12-2 = 10,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -2 \\ 1 & 6 \end{vmatrix} = -(18+2) = -20$$

and $$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -3-2 = -5$$

Matrix of cofactors $= \begin{pmatrix} -19 & 35 & 9 \\ 2 & -5 & -2 \\ 10 & -20 & -5 \end{pmatrix}$. Hence AdjA$= \begin{pmatrix} -19 & 2 & 10 \\ 35 & -5 & -20 \\ 9 & -2 & -5 \end{pmatrix}$

$$\therefore\ A^{-1} = \frac{1}{|A|}.Adj\ A = \frac{1}{-5}\begin{pmatrix} -19 & 2 & 10 \\ 35 & -5 & -20 \\ 9 & -2 & -5 \end{pmatrix} = \begin{bmatrix} \frac{19}{5} & \frac{-2}{5} & -2 \\ -7 & 1 & 4 \\ \frac{-9}{5} & \frac{2}{5} & 1 \end{bmatrix}$$

Note : Verify that $A^{-1}A = AA^{-1} = I_3$

Example 22.10 If $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$; find

(i) $(AB)^{-1}$ (ii) $B^{-1}A^{-1}$ (iii) Is $(AB)^{-1} = B^{-1}A^{-1}$?

Solution : (i) Here, $AB = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -2+0 & 1+0 \\ -4+0 & 2+1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$$

$\therefore \qquad |AB| = \begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = -6+4 = -2 \neq 0.$

Thus, $(AB)^{-1}$ exists.

Let us denote AB by C_{ij}

Let C_{ij} be the cofactor of the element C_{ij} of $|C|$.

Then, $C_{11} = (-1)^{1+1}(3) = 3 \qquad C_{21} = (-1)^{2+1}(1) = -1$

$C_{12} = (-1)^{1+2}(-4) = 4 \qquad C_{22} = (-1)^{2+2}(-2) = -2$

Hence, $\text{Adj}(C) = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$

$$C^{-1} = \frac{1}{|C|}\text{Adj}(C) = \frac{1}{-2}\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$C^{-1} = (AB)^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

(ii) To find $B^{-1}A^{-1}$, first we will find B^{-1}.

Now. $B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$ $\therefore$ $|B| = \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2-0 = 2 \neq 0$

$\therefore$ B^{-1} exists.

Let B_{ij} be the cofactor of the element bij of |B|

then $B_{11} = (-1)^{1+1}(-1) = -1$ $\quad B_{21} = (-1)^{2+1}(1) = -1$

$B_{12} = (-1)^{1+2}(0) = 0$ and $\quad B_{22} = (-1)^{2+2}(-2) = -2$

Hence, Adj $B = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$

$$\therefore \quad B^{-1} = \frac{1}{|B|}.AdjB = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}$$

Also, $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ Therefore, $|A| = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = 1 - 0 = -1 \quad \neq 0$

Therefore, A^{-1} exists.

Let A_{ij} be the cofactor of the element a_{ij} of |A|

then $A_{11} = (-1)^{1+1}(-1) = -1$ $\quad A_{21} = (-1)^{2+1}(0) = 0$

$A_{12} = (-1)^{1+2}(2) = -2$ and $\quad A_{22} = (-1)^{2+2}(1) = 1$

Hence, Adj $A = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} Adj\ A = \frac{1}{-1}\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

Thus, $$B^{-1}A^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2}-1 & 0+\frac{1}{2} \\ 0-2 & 0+1 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

(iii) From (i) and (ii), we find that

$$(AB)^{-1} = B^{-1}A^{-1} = = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

Hecne, $(AB)^{-1} = B^{-1}A^{-1}$

Notes

CHECK YOUR PROGRESS 22.3

1. Find, if possible, the inverse of each of the following matrices:

(a) $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

2. Find, if possible, the inverse of each of the following matrices :

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 & 2 \\ 5 & 2 & 4 \\ 1 & -3 & -2 \end{bmatrix}$

Verify that $A^{-1}A = AA^{-1} =$ I for (a) and (b).

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

4. Find $(A')^{-1}$ if $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$

show that ABA^{-1} is a diagonal matrix.

6. If $\phi(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $[\phi(x)]^{-1} = \phi(-x)$.

7. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

8. If $A = \begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$, show that $aA^{-1} = (a^2 + bc + 1)\,I - aA$

9. If $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = A^2$

10. If $A = \dfrac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & -4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, show that $A^{-1} = A'$

22.5 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

In earlier classes, you have learnt how to solve linear equations in two or three unknowns (simultaneous equations). In solving such systems of equations, you used the process of elimination of variables. When the number of variables invovled is large, such elimination process becomes tedious.

You have already learnt an alternative method, called Cramer's Rule for solving such systems of linear equations.

We will now illustrate another method called the matrix method, which can be used to solve the system of equations in large number of unknowns. For simplicity the illustrations will be for system of equations in two or three unknowns.

22.5.1 MATRIX METHOD

In this method, we first express the given system of equation in the matrix form $AX = B$, where A is called the co-efficient matrix.

For example, if the given system of equation is $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, we express them in the matrix equation form as :

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

If the given system of equations is $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$, then this system is expressed in the matrix equation form as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Where, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Before proceding to find the solution, we check whether the coefficient matrix A is non-singular or not.

Note: If A is singular, then $|A|=0$. Hence, A^{-1} does not exist and so, this method does not work.

Consider equation $AX = B$, where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

When $|A| \neq 0$, i.e. when $a_1b_2 - a_2b_1 \neq 0$, we multiply the equation $AX = B$ with A^{-1} on both side and get

$$A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow \quad (A^{-1}A)\,X = A^{-1}B$$

$$\Rightarrow \quad IX = A^{-1}B \qquad (\because A^{-1}A = I)$$

$$\Rightarrow \quad X = A^{-1}B$$

Since $A^{-1} = \dfrac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}$, we get

$$X = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2c_1 - b_1c_2 \\ -a_2c_1 + a_1c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \\ \dfrac{-a_2c_1 + a_1c_2}{a_1b_2 - a_2b_1} \end{bmatrix}$$

Hence, $x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$ and $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

Example 22.11 Using matrix method, solve the given system of linear equations.

$$\left.\begin{array}{l} 4x-3y=11 \\ 3x+7y=-1 \end{array}\right\} \qquad \text{......(i)}$$

Solution: This system can be expressed in the matrix equation form as

$$\begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad \text{......(ii)}$$

Here, $A = \begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$

so, (ii) reduces to

$AX = B$ ......(iii)

Now, $|A| = \begin{vmatrix} 4 & -3 \\ 3 & 7 \end{vmatrix} = 28 + 9 = 37 \neq 0$

Since $|A| \neq 0$, A^{-1} exists.

Now, on multiplying the equation $AX = B$ with A^{-1} on both sides, we get

$A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

i.e. $IX = A^{-1}B$

$X = A^{-1}B$

Hence, $X = \dfrac{1}{|A|}(Adj\ A)\ B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 7 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 11 \\ -1 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 77-3 \\ -33-4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 74 \\ -37 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So, $x = 2$, $y = -1$ is unique solution of the system of equations.

Example 22.12 Solve the following system of equations, using matrix method.

$$\left.\begin{array}{r} x+2y+3z=14 \\ x-2y+z=0 \\ 2x+3y-z=5 \end{array}\right\}$$

Solution : The given equations expressed in the matrix equation form as :

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix} \qquad \text{... (i)}$$

which is in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix}$$

$\therefore \quad X = A^{-1} B \qquad$ ... (ii)

Here, $\quad |A| = 1\,(2-3) \ -2\,(-1-2) +3\,(3+4)$

$\quad = 26 \neq 0$

$\therefore \quad A^{-1}$ exists.

Also, Adj $\quad A = \begin{bmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{bmatrix}$

Hence, from (ii), we have $\quad X = A^{-1}B = \dfrac{1}{|A|} AdjA.\,B$

$$X = \frac{1}{26}\begin{bmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{bmatrix}\begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 26 \\ 52 \\ 78 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus, $x = 1$, $y = 2$ and $z = 3$ is the solution of the given system of equations.

22.6 CRITERION FOR CONSISTENCY OF A SYSTEM OF EQUATIONS

Let $AX = B$ be a system of two or three linear equations.

Then, we have the following criteria :

(1) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution, given by $X = A^{-1}B$.

(2) If $|A| = 0$, then the system may or may not be consistent and if consistent, it does not have a unique solution. If in addition,

(a) $(\text{Adj A})\, B \neq O$, then the system is inconsistent.

(b) $(\text{Adj A})\, B = O$, then the system is consistent and has infinitely many solutions.

Note : These criteria are true for a system of *'n'* equations in *'n'* variables as well.

We now, verify these with the help of the examples and find their solutions wherever possible.

(a) $$\begin{aligned} 5x + 7y &= 1 \\ 2x - 3y &= 3 \end{aligned}$$

This system is consistent and has a unique solution, because $\begin{vmatrix} 5 & 7 \\ 2 & -3 \end{vmatrix} \neq 0$ Here, the matrix equation is $\begin{pmatrix} 5 & 7 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

i.e. $\qquad AX = B$ .... (i)

where, $A = \begin{pmatrix} 5 & 7 \\ 2 & -3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Here, $|A| = 5 \times (-3) - 2 \times 7 = -15 - 14 = -29 \neq 0$

$$\text{and } A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{-29}\begin{pmatrix} -3 & -7 \\ -2 & 5 \end{pmatrix} \qquad \text{.... (ii)}$$

From (i), we have $X = A^{-1}B$

i.e., $$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-29}\begin{pmatrix} -3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \dfrac{24}{29} \\ -\dfrac{13}{29} \end{pmatrix} \quad \text{[From (i) and (ii)]}$$

Thus, $x = \dfrac{24}{29}$, and $y = \dfrac{-13}{29}$ is the unique solution of the given system of equations.

(b) $$\begin{aligned} 3x + 2y &= 7 \\ 6x + 4y &= 8 \end{aligned}$$

In the matrix form the system can be written as

$$\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

or, $AX = B$

where $A = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$

Here, $|A| = 3 \times 4 - 6 \times 2 = 12 - 12 = 0$

$$\text{Adj } A = \begin{pmatrix} 4 & -6 \\ -6 & 3 \end{pmatrix}$$

Also, $(\text{Adj } A)\, B = \begin{bmatrix} 4 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} -20 \\ -18 \end{bmatrix} \neq 0$

Thus, the given system of equations is inconsistent.

(c) $$\left.\begin{aligned} 3x - y &= 7 \\ 9x - 3y &= 21 \end{aligned}\right\}$$

In the matrix form the system can be written as

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \end{pmatrix}$$

or, $AX = B$, where

$$A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}; \; X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 \\ 21 \end{pmatrix}$$

Here, $|A| = \begin{vmatrix} 3 & -1 \\ 9 & -3 \end{vmatrix} = 3 \times (-3) - 9 \times (-1) = -9 + 9 = 0$

$$\text{Adj A} = \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix}$$

Also, $(\text{Adj A})B = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

$\therefore$ The given system has an infinite number of solutions.

Let us now consider another system of linear equations, where $|A| = 0$ and $(\text{Adj A})\,B \neq O$. Consider the following system of equations

$$x + 2y + z = 5$$

$$2x + y + 2z = -1$$

$$x - 3y + z = 6$$

In matrix equation form, the above system of equations can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$$

i.e., $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0 \qquad (\because C_1 = C_3)$

Also, $(\text{Adj A})\,B = \begin{bmatrix} 7 & -5 & 3 \\ 0 & 0 & 0 \\ -7 & 5 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$ [Verify (Adj A) yourself]

$$= \begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix} \neq O$$

Since $|A| = 0$ and $(\text{Adj A})\,B \neq O$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}(Adj \;\; A)B$$

$$= \frac{1}{0}\begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix} \text{ which is undefined.}$$

The given system of linear equation will have no solution.

Thus, we find that if $|A| = 0$ and (Adj A) $B \neq$ O then the system of equations will have no solution.

We can summarise the above findings as:

(i) If $|A| \neq 0$ and (Adj A) $B \neq$ O then the system of equations will have a non-zero, unique solution.

(ii) If $|A| \neq 0$ and (Adj A) B = O, then the system of equations will have trivial solutions.

(iii) If $|A| = 0$ and (Adj A) B = O, then the system of equations will have infinitely many solutions.

(iv) If $|A| = 0$ and (Adj A) $B \neq$ O, then the system of equations will have no solution Inconsistent.

CHECK YOUR PROGRESS 22.4

1. Solve the following system of equations, using the matrix inversion method:

(a) $2x + 3y = 4$
$x - 2y = 5$

(b) $x + y = 7$
$3x - 7y = 11$

2. Solve the following system of equations using matrix inversion method:

(a) $x + 2y + z = 3$
$2x - y + 3z = 5$
$x + y - z = 7$

(b) $2x + 3y + z = 13$
$3x + 2y - z = 12$
$x + y + 2z = 5$

(c) $-x + 2y + 5z = 2$
$2x - 3y + z = 15$
$-x + y + z = -3$

(d) $2x + y - z = 2$
$x + 2y - 3z = -1$
$5x - y - 2z = -1$

3. Determine whether the following system of equations are consistent or not. If consistent, find the solution:

(a) $2x - 3y = 5$

$x + y = 7$

(b) $2x - 3y = 5$

$4x - 6y = 10$

(c) $3x + y + 2z = 3$

$-2y - z = 7$

$x + 15y + 3z = 11$

LET US SUM UP

- A square matrix is said to be non-singular if its corresponding determinant is non-zero.
- The determinant of the matrix A obtained by deleting the i[th] row and j[th] column of A, is called the minor of a_{ij}. It is usually denoted by M_{ij}.
- The cofactor of a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$
- Adjoint of a matrix A is the transpose of the matrix whose elements are the cofactors of the elements of the determinat of given matrix. It is usually denoted by Adj A.
- If A is any square matrix of order n, then

  $A\,(\text{Adj}\,A) = (\text{Adj}\,A)\,A = |A|\ I_n$ where I_n is the unit matrix of order n.
- For a given non-singular square matrix A, if there exists a non-singular square matrix B such that $AB = BA = I$, then B is called the multiplicative inverse of A. It is written as $B = A^{-1}$.
- Only non-singular square matrices have multiplicative inverse.
- If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, then we can express the system in the matrix equation form as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, if $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$$X = A^{-1}B = \frac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- A system of equations, given by $AX = B$, is said to be consistent and has a unique solution, if $|A| \neq 0$.
- A system of equations, given by $AX = B$, is said to be inconsistent, if $|A| = 0$ and $(\text{Adj } A)\, B \neq O$.
- A system of equations, given by $AX = B$, is said be be consistent and has infinitely many solutions, if $|A| = 0$, and $(\text{Adj } A)\, B = O$.

SUPPORTIVE WEB SITES

http://www.mathsisfun.com/algebra/matrix-inverse.html

http://www.sosmath.com/matrix/coding/coding.html

TERMINAL EXERCISE

1. Find $|A|$, if

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ -3 & -1 & 0 \\ -2 & 5 & 4 \end{pmatrix}$ (b) $A = \begin{pmatrix} -1 & 3 & 4 \\ 7 & 5 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

2. Find the adjoint of A, if

(a) $A = \begin{pmatrix} -2 & 3 & 7 \\ -1 & 4 & 5 \\ -1 & 0 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 1 & 2 \\ -2 & 1 & 3 \end{pmatrix}$

Also, verify that $A(\text{Adj } A) = |A| I_3 = (\text{Adj } A) A$, for (a) and (b)

3. Find A^{-1}, if exists, when

(a) $\begin{pmatrix} 3 & 6 \\ 7 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}$

Also, verify that $(A')^{-1} = (A^{-1})'$, for (a), (b) and (c)

4. Find the inverse of the matrix A, if

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

5. Solve, using matrix inversion method, the following systems of linear equations

(a) $\begin{aligned} x + 2y &= 4 \\ 2x + 5y &= 9 \end{aligned}$ (b) $\begin{aligned} 6x + 4y &= 2 \\ 9x + 6y &= 3 \end{aligned}$

(c) $\begin{aligned} 2x + y + z &= 1 \\ x - 2y - z &= \frac{3}{2} \\ 3y - 5z &= 9 \end{aligned}$ (d) $\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2 \end{aligned}$

(e) $\begin{aligned} x + y - 2z &= -1 \\ 3x - 2y + z &= 3 \\ 2x + y - z &= 0 \end{aligned}$

6. Solve, using matrix inversion method

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4; \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1; \quad \frac{8}{x} + \frac{9}{y} - \frac{20}{z} = 3$$

7. Find the value of λ for which the following system of equation becomes consistent

$$\begin{aligned} 2x - 3y + 4 &= 0 \\ 5x - 2y - 1 &= 0 \\ 21x - 8y + \lambda &= 0 \end{aligned}$$

ANSWERS

CHECK YOUR PROGRESS 22.1

1. (a) –12 (b) 10
2. (a) singular (b) non-singular
3. (a) $M_{11} = 4;\ M_{12} = 7;\ M_{21} = -1;\ M_{22} = 3$ (b) $M_{11} = 5;\ M_{12} = 2;\ M_{21} = 6;\ M_{22} = 0$
4. (a) $M_{21} = 11;\ M_{22} = 7;\ M_{23} = 1$ (b) $M_{31} = -13;\ M_{32} = -13;\ M_{33} = 13$
5. (a) $C_{11} = 7;\ C_{12} = -9;\ C_{21} = 2;\ C_{22} = 3$ (b) $C_{11} = 6;\ C_{12} = 5;\ C_{21} = -4;\ C_{22} = 0$
6. (a) $C_{21} = 1;\ C_{22} = -8;\ C_{23} = -2$ (b) $C_{11} = -6;\ C_{12} = 10;\ C_{33} = 2$

CHECK YOUR PROGRESS 22.2

1. (a) $\begin{bmatrix} 6 & 1 \\ -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & i \\ -i & i \end{bmatrix}$

CHECK YOUR PROGRESS 22.3

1. (a) $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{4}{10} & -\frac{2}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

2. (a) $\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \\ -\frac{8}{5} & \frac{6}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{7}{12} & \frac{1}{3} & \frac{1}{12} \\ \frac{17}{24} & -\frac{1}{3} & -\frac{11}{24} \end{bmatrix}$

4. $(A')^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

CHECK YOUR PROGRESS 22.4

1. (a) $x = \frac{23}{7},\ y = \frac{-6}{7}$ (b) $x = 6, y = 1$

2. (a) $x = \frac{58}{11},\ y = -\frac{2}{11},\ z = -\frac{21}{11}$ (b) $x = 2,\ y = 3,\ z = 0$

(c) $x = 2,\ y = -3,\ z = 2$ (d) $x = 1,\ y = 2,\ z = 2$

3. (a) Consistent; $x = \frac{26}{5},\ y = \frac{9}{5}$ (b) Consistent; infinitely many solutions

(c) Inconsistent

TERMINAL EXERCISE

1. (a) –31 (b) –24

2. (a) $\begin{bmatrix} 4 & -3 & -13 \\ -4 & 5 & 3 \\ 4 & -3 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 8 & -7 \\ -13 & 13 & 13 \\ 5 & 1 & 4 \end{bmatrix}$

3. (a) $\begin{bmatrix} -\frac{1}{18} & \frac{1}{6} \\ \frac{7}{36} & -\frac{1}{12} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{5}{7} & -\frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix}$ (c) $\begin{bmatrix} -\frac{1}{7} & -\frac{5}{14} \\ -\frac{2}{7} & -\frac{3}{14} \end{bmatrix}$

4. (a) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{3} & 0 \\ 3 & \frac{2}{3} & -1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

5. (a) $x = 2,\ y = 1$ (b) $x = k,\ y = \frac{1}{2} - \frac{3}{2}k$

(c) $x = 1,\ y = \frac{1}{2},\ z = -\frac{3}{2}$ (d) $x = 2,\ y = -1,\ z = 1$

(e) $x = \frac{1}{2},\ y = -\frac{1}{2},\ z = \frac{1}{2}$

6. $x = 2,\ y = 3,\ z = 5$

7. $\lambda = -5$

23

RELATIONS AND FUNCTIONS-II

We have learnt about the basic concept of Relations and Functions. We know about the ordered pair, the cartesian product of sets, relation, functions, their domain, Co-doman and range. Now we will extend our knowledge to types of relations and functions, composition of functions, invertible functions and binary operations.

OBJECTIVES

After studying this lesson, you will be able to :

- verify the equivalence relation in a set
- verify that the given function is one-one, many one, onto/ into or one one onto
- find the inverse of a given function
- determine whether a given operation is binary or not.
- check the commutativity and associativity of a binary operation.
- find the inverse of an element and identity element in a set with respest to a binary operation.

EXPECTED BACKGROUND KNOWLEDGE

Before studying this lesson, you should know :

- Concept of set, types of sets, operations on sets
- Concept of ordered pair and cartesian product of set.
- Domain, co-domain and range of a relation and a function

23.1 RELATION

23.1.1 Relation :

Let A and B be two sets. Then a relation R from Set A into Set B is a subset of A × B.

Thus, R is a relation from A to B $\Leftrightarrow$ R $\subseteq$ A × B

- If $(a, b) \in$ R then we write aRb which is read as 'a' is related to b by the relation R, if $(a, b) \notin$ R, then we write $a \not\mathrel{R} b$ and we say that a is not related to b by the relation R.
- If $n(A) = m$ and $n(B) = n$, then A × B has mn ordered pairs, therefore, total number of relations form A to B is 2^{mn}.

23.1.2 Types of Relations

(i) Reflexive Relation :

A relation R on a set A is said to be reflexive if every element of A is related to itself.
Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$
A relation R is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Let A = {1, 2, 3} be a set. Then

R = {(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)} is a reflexive relation on A.

but R_1 = {(1, 1), (3, 3) (2, 1) (3, 2)} is not a reflexive relation on A, because $2 \in A$ but $(2,2) \notin R$.

(ii) Symmetric Relation

A relation R on a set A is said to be symmetric relation if

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $(a, b) \in A$

i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.

Let A = {1, 2, 3, 4} and R_1 and R_2 be relations on A given by

R_1 = {(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)}

and R_2 = {(1, 1), (2, 2), (3, 3), (1, 3)}

- R_1 is symmetric relation on A because $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$

  or $aR_1b \Rightarrow bR_1 a$ for all $a,b \in A$

  but R_2 is not symmetric because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$.

A reflexive relation on a set A is not necessarily symmetric. For example, the relation R ={(1, 1), (2, 2), (3, 3), (1, 3)} is a reflexive relation on set A = {1, 2, 3} but it is not symmetric.

(iii) Transitive Relation:

Let A be any set. A relation R on A is said to be transitive relation if

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$

For example :

On the set N of natural numbers, the relation R defined by xRy

$\Rightarrow$ 'x is less than y', is transitive, because for any $x, y, z \in N$

$x < y$ and $y < z \Rightarrow x < z$

i.e. xRy and $yRz \Rightarrow xRz$

Take another example

Let A be the set of all straight lines in a plane. Then the relation 'is parallel to' on A is a transitive relation, because for any $l_1, l_2, l_3 \in A$

$l_1 \parallel l_2$ and $l_2 \parallel l_3 \Rightarrow l_1 \parallel l_3$

Example 23.1 Check the relation R for reflexivity, symmetry and transitivity, where R is defined as $l_1 R l_2$ iff $l_1 \perp l_2$ for all $l_1, l_2 \in A$

Solution : Let A be the set of all lines in a plane. Given that $l_1 R l_2 \Leftrightarrow l_1 \perp l_2$ for all $l_1, l_2 \in A$

Reflexivity : R is not reflexive because a line cannot be perpendicular to itself i.e. $l \perp l$ is not true.

Symmetry : Let $l_1, l_2 \in A$ such that $l_1 R l_2$

Then $l_1 \, R l_2 \Rightarrow l_1 \perp l_2 \Rightarrow l_2 \perp l_1 \Rightarrow l_2 R l_1$

So, R is symmetric on A

Transitive

R is not transitive, because $l_1 \perp l_2$ and $l_2 \perp l_3$ does not impty that $l_1 \perp l_3$

23.2 EQUIVALENCE RELATION

A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

For example the relation 'is congruent to' is an equivalence relation because

(i) it is reflexive as $\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all $\Delta \in S$ where S is a set of triangles.

(ii) it is symmetric as $\Rightarrow \Delta_1 R \Delta_2 \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1$

$$\Rightarrow \Delta_2 R \Delta_1$$

(iii) it is transitive as $\Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3$

it means $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow (\Delta_1, \Delta_3) \in R$

Example 23.2 Show that the relation R defined on the set A of all triangles in a plane as $R = \{(T_1, T_2) : T_1$ is similar to $T_2)$ is an equivalence relation.

Solution : We observe the following properties of relation R;

Reflexivity we know that every triangle is similar to itself. Therefore, $(T, T) \in R$ for all $T \in A \Rightarrow R$ is reflexive.

Symmetricity Let $(T_1, T_2) \in R$, then

$(T_1, T_2) \in R \quad \Rightarrow T_1$ is similar to T_2

$\Rightarrow T_2$ is similar to T_1

$\Rightarrow (T_2, T_1) \in R$, So, R is symmetric.

Transitivity : Let $T_1, T_2, T_3 \in A$ such that $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$.

Then $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3

$\Rightarrow T_1$ is similar to T_3

$\Rightarrow (T_1, T_3) \in R$

Hence, R is an equivalence relation.

CHECK YOUR PROGRESS 23.1

1. Let R be a relation on the set of all lines in a plane defined by $(l_1, l_2) \in R \Rightarrow$ line l_1 is parallel to l_2. Show that R is an equivalence relation.
2. Show that the relation R on the set A of points in a plane, given by

   R = {(P, Q) : Distance of the point P from the origin is same as the distance of the point Q from the origin} is an equivalence relation.
3. Show that each of the relation R in the set $A = \{x \in z : 0 \le x \le 12\}$, given by

   (i) $R = \{(a,b) : |a-b| \, is \, multiple \, of \; 4\}$

   (ii) $R = \{(a,b) : a = b\}$ is an equivalence relation
4. Prove that the relation 'is a factor of' from R to R is reflexive and transitive but not symmetric.
5. If R and S are two equivalence relations on a set A then $R \cap S$ is also an equivalence relation.
6. Prove that the relation R on set $N \times N$ defined by (a,b) R (c,d) $\Leftrightarrow$ a+d = b + c for all (a,b), (c,d) $\in N \times N$ is an equivalence relation.

23.3 CLASSIFICATION OF FUNCTIONS

Let f be a function from A to B. If every element of the set B is the image of at least one element of the set A i.e. if there is no unpaired element in the set B then we say that the **function f maps the set *A* onto the set *B*. Otherwise we say that the function maps the set *A* into the set *B*.**

Functions for which each element of the set A is mapped to a different element of the set B are said to be ***one-to-one.***

One-to-one function

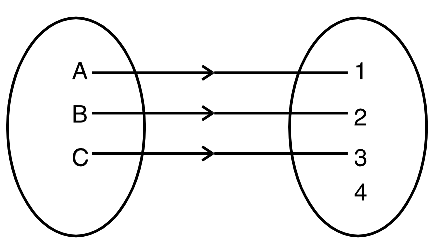


Fig.23.27

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 2, 3\}$

A function can map more than one element of the set A to the same element of the set B. Such a type of function is said to be ***many-to-one.***

Many-to-one function

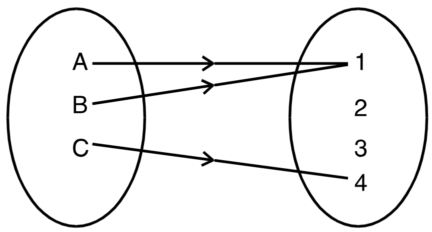


Fig. 23.2

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 4\}$

A function which is both one-to-one and onto is said to be a bijective function.

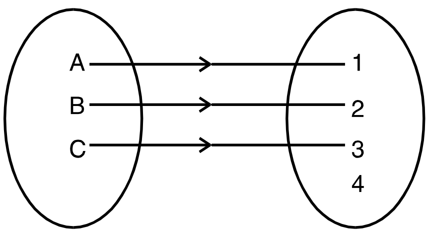


Fig. 23.3

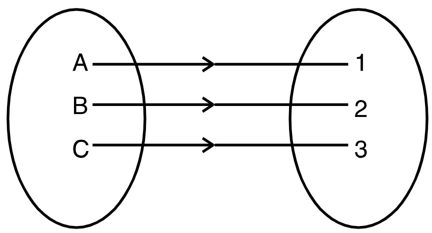


Fig. 23.4

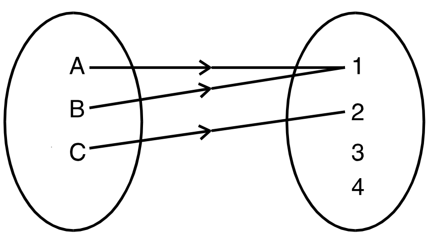


Fig. 23.5

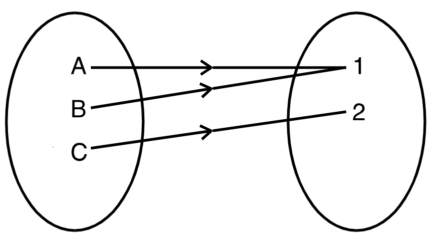


Fig. 23.6

Fig. 23.3 shows a one-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 23.4 shows a one-to-one function mapping $\{A, B, C\}$ onto $\{1, 2, 3\}$.

Fig. 23.5shows a many-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 23.6 shows a many-to-one function mapping $\{A, B, C\}$ onto $\{1, 2\}$.

Function shown in Fig. 23.4 is also a bijective Function.

Note : Relations which are one-to-many can occur, but they are not functions. The following figure illustrates this fact.

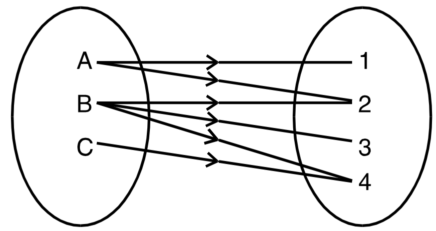


Fig. 23.7

Example 23.3 Without using graph prove that the function

$$f : R \to R \text{ defiend by } f(x) = 4 + 3x \text{ is } \textbf{\textit{one-to-one}}.$$

Solution : For a function to be one-one function

$$f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2 \quad \forall \quad x_1, x_2 \in \text{domain}$$

$\therefore$ Now $f(x_1) = f(x_2)$ gives

$$4 + 3x_1 = 4 + 3x_2 \quad \text{or} \quad x_1 = x_2$$

$\therefore$ f is a ***one-one function***.

Example 23.4 Prove that

$$f : R \to R \text{ defined by } f(x) = 4x^3 - 5 \text{ is a bijection}$$

Solution : Now $f(x_1) = f(x_2) \quad \forall \quad x_1, x_2 \in \text{Domain}$

$$\therefore \quad 4x_1^3 - 5 = 4x_2^3 - 5$$

$$\Rightarrow \quad x_1^3 = x_2^3$$

$$\Rightarrow \quad x_1^3 - x_2^3 = 0 \Rightarrow \quad (x_2 - x_1)\left(x_1^2 + x_1x_2 + x_2^2\right) = 0$$

$$\Rightarrow \quad x_1 = x_2 \text{ or}$$

$x_1^2 + x_1x_2 + x_2^2 = 0$ (rejected). It has no real value of x_1 and x_2.

$\therefore$ f is a ***one-one function***.

Again let $y = (x)$ where $y \in$ codomain, $x \in$ domain.

We have $y = 4x^3 - 5$ or $x = \left(\dfrac{y+5}{4}\right)^{1/3}$

$\therefore$ For each $y \in$ codomain $\exists\, x \in$ domain such that $f(x) = y$.

Thus f is ***onto function***.

$\therefore$ f is a bijection.

Example 23.5 Prove that $f : R \to R$ defined by $f(x) = x^2 + 3$ is neither ***one-one*** nor ***onto function***.

Solution : We have $f(x_1) = f(x_2)\ \forall\ x_1, x_2 \in$ domain giving

$$x_1^2 + 3 = x_2^2 + 3 \Rightarrow x_1^2 = x_2^2$$

or $\quad x_1^2 - x_2^2 = 0 \Rightarrow x_1 = x_2 \quad$ or $\quad x_1 = -x_2$

or $\quad$ f is not ***one-one function***.

Again let $y = f(x)$ where $y \in$ codomain

$$x \in \text{domain.}$$

$\Rightarrow \quad y = x^2 + 3 \quad \Rightarrow \quad x = \pm\sqrt{y-3}$

$\Rightarrow \quad \forall\ y < 3\ \exists$ no real value of x in the domain.

$\therefore \quad$ f is not an ***onto finction***.

23.4 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

$$y = x^2$$

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

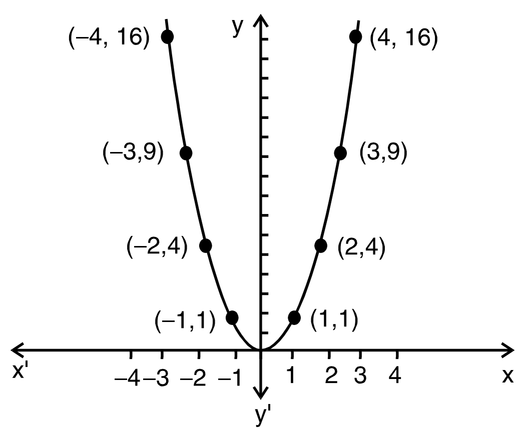


Fig. 23.8

Does this represent a function?

Yes, this represent a function because corresponding to each value of $x\ \exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

$$x^2 + y^2 = 25$$

x	0	0	3	3	4	4	5	–5	–3	–3	–4	–4
y	5	–5	4	–4	3	–3	0	0	4	–4	3	–3

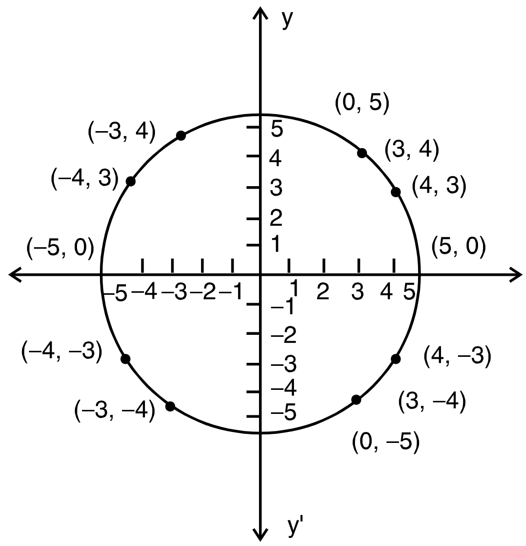


Fig. 23.9

This graph represents a circle.

Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

CHECK YOUR PROGRESS 23.2

1. (i) Does the graph represent a function?

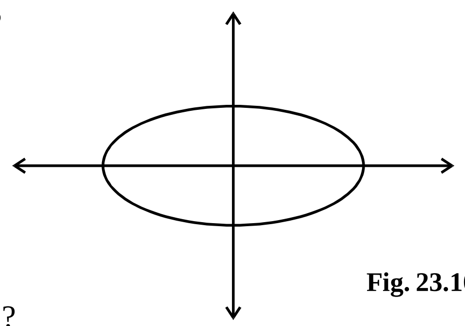


Fig. 23.10

(ii) Does the graph represent a function ?

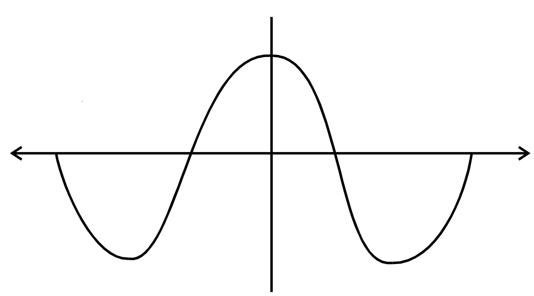


Fig. 23.11

2. Which of the following functions are into function ?

(a)

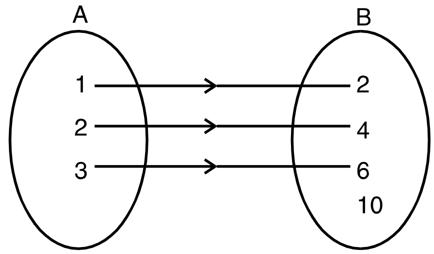


Fig.23.12

(b) $f : N \to N$, defined as $f(x) = x^2$

Here N represents the set of natural numbers.

(c) $f : N \to N$, defined as $f(x) = x$

3. Which of the following functions are onto function if $f : R \to R$

(a) $f(x) = 115x + 49$ (b) $f(x) = |x|$

4. Which of the following functions are one-to-one functions ?

(a) $f : \{20, 21, 22\} \to \{40, 42, 44\}$ defined as $f(x) = 2x$

(b) $f : \{7, 8, 9\} \to \{10\}$ defined as $f(x) = 10$

(c) $f : I \to R$ defined as $f(x) = x^3$

(d) $f : R \to R$ defined as $f(x) = 2 + x^4$

(d) $f : N \to N$ defined as $f(x) = x^2 + 2x$

5. Which of the following functions are many-to-one functions ?

(a) $f : \{-2, -1, 1, 2\} \to \{2, 5\}$ defined as $f(x) = x^2 + 1$

(b) $f : \{0, 1, 2\} \to \{1\}$ defined as $f(x) = 1$

(c)

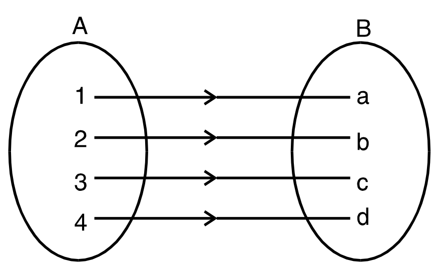


Fig.23.13

(d) $f : N \to N$ defined as $f(x) = 5x + 7$

MODULE - VII
Relation and Function

Notes

23.5 COMPOSITION OF FUNCTIONS

Consider the two functions given below:

$$y = 2x + 1, \qquad x \in \{1, 2, 3\}$$

$$z = y + 1, \qquad y \in \{3, 5, 7\}$$

Then *z* is the composition of two functions *x* and *y* because *z* is defined in terms of *y* and *y* in terms of *x*.

Graphically one can represent this as given below :

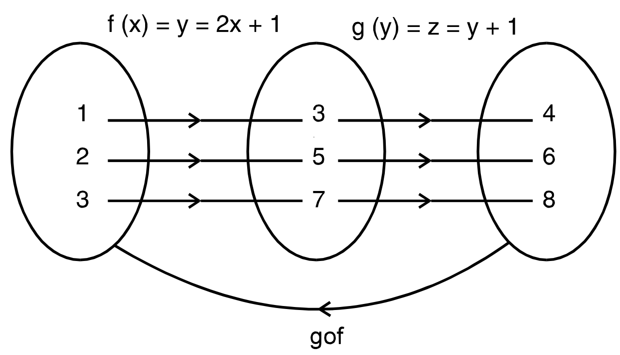


Fig. 23.18

The composition, say, gof of function g and f is defined as function g of function f.

If $\quad f : A \to B$ and $g : B \to C$

then $\quad g \circ f : A$ to C

Let $\quad f(x) = 3x + 1$ and $\quad g(x) = x^2 + 2$

Then $\quad fog(x) = f(g(x)) = f(x^2 + 2)$

$$= 3(x^2 + 2) + 1 = 3x^2 + 7 \qquad \text{(i)}$$

and $\quad (gof)(x) = g(f(x)) = g(3x + 1)$

$$= (3x + 1)^2 + 2 = 9x^2 + 6x + 3 \qquad \text{(ii)}$$

Check from (i) and (ii), if

$$fog = gof$$

Evidently, $\quad fog \neq gof$

Similarly, $(fof)(x) = f(f(x)) = f(3x + 1)$ [Read as function of function f].

$$= 3(3x + 1) + 1 = 9x + 3 + 1 = 9x + 4$$

$(gog)(x) = g(g(x)) = g(x^2 + 2)$ [Read as function of function g]

$$= (x^2 + 2)^2 + 2 = x^4 + 4x^2 + 4 + 2 = x^4 + 4x^2 + 6$$

Example 23. 6 If $f(x) = \sqrt{x+1}$ and $g(x) = x^2 + 2$, calculate *fog* and *gof*.

Solution :

$$fog(x) = f(g(x))$$

$$= f\left(x^2+2\right) = \sqrt{x^2+2+1} = \sqrt{x^2+3}$$

$$(gof)(x) = g(f(x))$$

$$= g\left(\sqrt{x+1}\right) = \left(\sqrt{x+1}\right)^2 + 2 = x+1+2 = x+3.$$

Here again, we see that $(fog) \neq gof$

Example 23. 7 If $f(x) = x^3$, $f : R \to R$ and $g(x) = \frac{1}{x}$, $g : R - \{0\} \to R - \{0\}$

Find *fog* and *gof*.

Solution : $(fog)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$

$$(gof)(x) = g(f(x)) = g\left(x^3\right) = \frac{1}{x^3}$$

Here we see that $fog = gof$

CHECK YOUR PROGRESS 23.3

1. Find *fog*, *gof*, *fof* and *gog* for the following functions :

   $f(x) = x^2 + 2$, $g(x) = 1 - \frac{1}{1-x}$, $x \neq 1$.

2. For each of the following functions write *fog*, *gof*, *fof* and *gog*.

   (a) $f(x) = x^2 - 4$, $g(x) = 2x + 5$

   (b) $f(x) = x^2$, $g(x) = 3$

   (c) $f(x) = 3x - 7$, $g(x) = \frac{2}{x}$, $x \neq 0$

3. Let $f(x) = |x|$, $g(x) = [x]$. Verify that $fog \neq gof$.

4. Let $f(x) = x^2 + 3$, $g(x) = x - 2$

   Prove that $fog \neq gof$ and $f\left(f\left(\frac{3}{2}\right)\right) = g\left(f\left(\frac{3}{2}\right)\right)$

5. If $f(x) = x^2$, $g(x) = \sqrt{x}$. Show that $fog = gof$.

6. Let $f(x) = |x|$, $g(x) = (x)^{\frac{1}{3}}$, $h(x) = \frac{1}{x}$; $x \neq 0$.

   Find (a) *fog* (b) *goh* (c) *foh* (d) *hog* (e) *fogoh*

23.6 INVERSE OF A FUNCTION

(A) Consider the relation

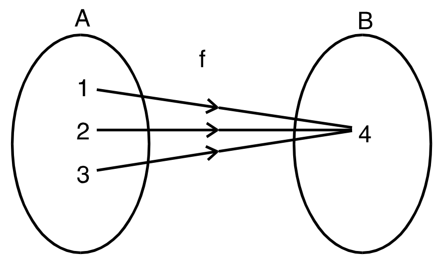


Fig. 23.19

This is a many-to-one function. Now let us find the inverse of this relation. Pictorially, it can be represented as

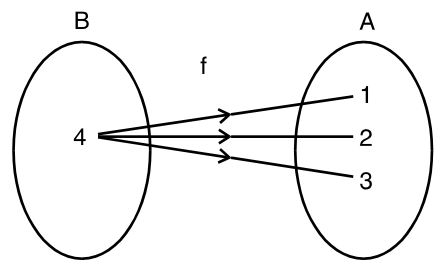


Fig 23.20

Clearly this relation does not represent a function. (Why ?)

(B) Now take another relation

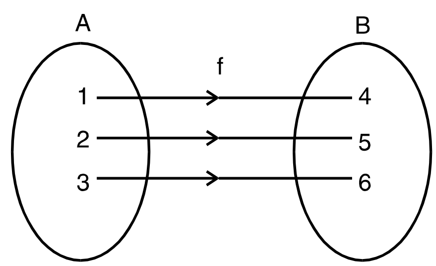


Fig.23.21

It represents one-to-one onto function. Now let us find the inverse of this relation, which is represented pictorially as

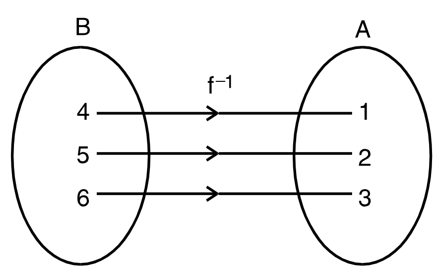


Fig. 23.22

This represents a function. (C) Consider the relation

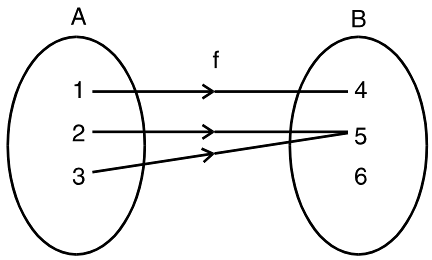


Fig. 23.23

Ir represents many-to-one function. Now find the inverse of the relation.

Pictorially it is represented as

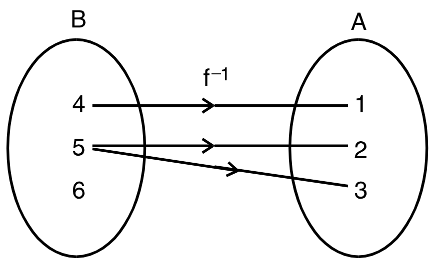


Fig. 23.24

This does not represent a function, because element 6 of set B is not associated with any element of A. Also note that the elements of B does not have a unique image.

(D) Let us take the following relation

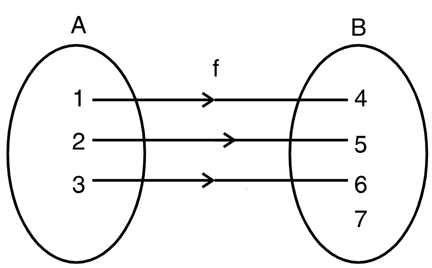


Fig. 23.25

It represent one-to-one into function. Find the inverse of the relation.

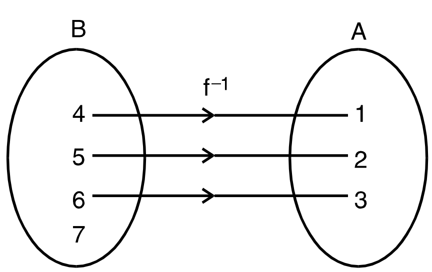


Fig. 23.26

It does not represent a function because the element 7 of *B* is not associated with any element of *A*. From the above relations we see that we may or may not get a relation as a function when we find the inverse of a relation (function).

We see that the inverse of a function exists only if the function is one-to-one onto function i.e. only if it is a bijective function.

CHECK YOUR PROGRESS 23.4

1 (i) Show that the inverse of the function

$$y = 4x - 7 \text{ exists.}$$

(ii) Let f be a one-to-one and onto function with domain A and range B. Write the domain and range of its inverse function.

2. Find the inverse of each of the following functions (if it exists) :

(a) $f(x) = x + 3 \quad \forall\ x \in R$

(b) $f(x) = 1 - 3x \quad \forall\ x \in R$

(c) $f(x) = x^2 \quad \forall\ x \in R$

(d) $f(x) = \dfrac{x+1}{x},\ x \neq 0 \quad x \in R$

23.7 BINARY OPERATIONS :

Let A, B be two non-empty sets, then a function from A × A to A is called a binary operation on A.

If a binary operation on A is denoted by '*', the unique element of A associated with the ordered pair (*a*, *b*) of A × A is denoted by *a* * *b*.

The order of the elements is taken into consideration, i.e. the elements associated with the pairs (*a*, *b*) and (*b*, *a*) may be different i.e. *a* * *b* may not be equal to *b* * *a*.

Let A be a non-empty set and '*' be an operation on A, then

1. A is said to be closed under the operation * iff for all $a, b \in A$ implies $a * b \in A$.
2. The operation is said to be commutative iff $a * b = b * a$ for all $a, b \in A$.
3. The operation is said to be associative iff $(a * b) * c = a * (b * c)$ for all $a, b, c \in A$.
4. An element $e \in A$ is said to be an identity element iff $e * a = a = a * e$
5. An element $a \in A$ is called invertible iff these exists some $b \in A$ such that $a * b = e = b * a$, b is called inverse of a.

Note : If a non empty set A is closed under the operation *, then operation * is called a binary operation on A.

For example, let A be the set of all positive real numbers and '*' be an operation on A defined by $a * b = \frac{ab}{3}$ for all $a, b \in$ A

For all $a, b, c \in$ A, we have

(i) $a * b = \frac{ab}{3}$ is a positive real number $\Rightarrow$ A is closed under the given operation.

$\therefore$ * is a binary operation on A.

(ii) $a*b = \frac{ab}{3} = \frac{ba}{3} = b*a \Rightarrow$ the operation * is commutative.

(iii) $(a*b)*c = \frac{ab}{3}*c = \frac{\frac{ab}{3}.c}{3} = \frac{abc}{9}$ and $a*(b*c) = a*\frac{bc}{3} = \frac{a}{3}.\frac{bc}{3} = \frac{abc}{9}$ –

$\Rightarrow (a*b)*c = a*(b*c) \Rightarrow$ the operation * is associative.

(iv) There exists $3 \in$ A such that $3 * a = 3.\frac{a}{3} = a = \frac{a}{3}.3 = a*3$

$\Rightarrow$ 3 is an identity element.

(v) For every $a \in$ A, there exists $\frac{9}{a} \in$ A such that $a*\frac{9}{a} = \frac{a.\frac{9}{a}}{3} = 3$ and $\frac{9}{a}*a = \frac{\frac{9}{a}.a}{3} = 3$

$\Rightarrow a*\frac{9}{a} = 3 = \frac{9}{a}*a \Rightarrow$ every element of A is invertible, and inverse of a is $\frac{9}{a}$

CHECK YOUR PROGRESS 23.5

1. Determine whether or not each of operation * defined below is a binary operation.

(i) $a*b = \frac{a+b}{2}, \forall a,b \in Z$

(ii) $a*b = a^b, \forall a,b \in Z$

(iii) $a*b = a^2 + 3b^2, \forall a,b \in R$

2. If $A = \{1,2\}$ find total number of binary operations on A.

3. Let a binary operation '*' on Q (set of all rational numbers) be defined as $a * b = a + 2b$ for all $a, b \in$Q.

Prove that

(i) The given operation is not commutative.

(ii) The given operation is not associative.

4. Let * be the binary operation difined on $Q^{+} by\, a * b = \dfrac{ab}{3}$ for all $a, b \in Q^{+}$ then find the inwrse of 4*6.

5. Let $A = N \times N$ and * be the binary operation on A defined by (a,b)*(c,d)=(a+c, b+d). Show that * is commutative and associative. Find the identity element of on A if any

6. A binary operation * on Q - {-1} is defined by a * b = a+b+ab; for all $a, b \in Q - \{-1\}$. Find identity element on Q. Also find the inverse of an element in Q-{-1}.

LET US SUM UP

- Reflexive relation R in X is a relation with $(a, a) \in$R $\forall$ $a \in$X.
- Symmetric relation R in X is a relation satisfying $(a, b) \in$R implies $(b, a) \in$R.
- Transitive relation R in X is a relation satisfying $(a, b) \in$R and $(b, c) \in$R implies that $(a, c) \in$R.
- Equivalence relation R in X is a relation which is reflexive, symmetric and transitive.
- If range is a subset of co-domain that function is called on into function.
- If f: $A \to B$, and f (x) = f (y) $\to$ x = y that function is called one-one function.
- Any function is inuertible if it is one-one-onto or bijective.
- If more than one element of A has only one image in to than function is called many one function .
- A binary operation * on a set A is a function * from A × A to A.
- If $a * b = b *$ a for all $a, b \in$A, then the operation is said to be commutative.
- If $(a * b) * c = a * (b * c)$ for all $a, b, \in$A, then the operation is said to be associative.
- If $e * a = a = a * e$ for all $a \in$A, then element $e \in$A is said to be an identity element.
- If $a * b = e = b * a$ then a and b are inverse of each other
- A pair of elements grouped together in a particular order is called an a ordered pair.
- If n(A) = p, n(B) = q then n(A × B) = pq
- R × R = $\{(x, y) : x, y \in$ R$\}$ and R × R × R = $\{(x, y, z) : x, y, z \in$R$\}$

- In a function $f : A \to B$, B is the codomain of f.
- $f, g : X \to R$ and $X \subset R$, then

  $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$

  $$(f \cdot g)x = f(x) \cdot g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

- A real function has the set of real number or one of its subsets both as its domain and as its range.

SUPPORTIVE WEBSITES

http://www.bbc.co.uk/education/asguru/maths/13pure/02functions/06composite/ index.shtml

http://mathworld.wolfram.com/Composition.html

http://www.cut-the-knot.org/Curriculum/Algebra/BinaryColorDevice.shtml

http://mathworld.wolfram.com/BinaryOperation.html

TERMINAL EXERCISE

1. Write for each of the following functions fog, gof, fof, gog.

   (a) $f(x) = x^3$ $\qquad$ $g(x) = 4x - 1$

   (b) $f(x) = \frac{1}{x^2}, x \neq 0$ $\qquad$ $g(x) = x^2 - 2x + 3$

   (c) $f(x) = \sqrt{x-4}, x \geq 4$ $\qquad$ $g(x) = x - 4$

   (d) $f(x) = x^2 - 1$ $\qquad$ $g(x) = x^2 + 1$

2. (a) Let $f(x) = |x|,\ g(x) = \frac{1}{x}, x \neq 0,\ h(x) = x^{\frac{1}{3}}$. Find fogoh

   (b) $f(x) = x^2 + 3,\ g(x) = 2x^2 + 1$

   Find fog (3) and gof (3).

3. Which of the following equations describe a function whose inverse exists :

   (a) $f(x) = |x|$ $\qquad$ (b) $f(x) = \sqrt{x}, x \geq 0$

   (c) $f(x) = x^2 - 1, x \geq 0$ $\qquad$ (d) $f(x) = \frac{3x-5}{4}$ $\qquad$ (e) $f(x) = \frac{3x+1}{x-1}$ $x \neq 1$.

4. If $gof(x) = |\sin x|$ and $gof(x) = \left(\sin\sqrt{x}\right)^2$ then find f (x) and g (x)

5. Let $*$ be a binary operation on Q defined by $a * b = \dfrac{a+b}{3}$ for all a,b $\in Q$, prove that $*$ is commutative on Q.

6. Let $*$ be a binary operation on on the set Q of rational numbers define by $a * b = \dfrac{ab}{5}$ for all a,b $\in Q$, show that $*$ is associative on Q.

7. Show that the relation R in the set of real numbers, defined as R = $\{(a, b)\} : a \leq b^2\}$ is neither reflexive, nor symmetric nor transitive.

8. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as R = $\{(a, b) : b = a + 1\}$ is reflexive, symmetric and transitive.

9. Show that the relation R in the set A defined as R = $\{(a, b)\ \forall : a = b\}\ a, b \in$A, is equivalence relation.

10. Let A = N × N, N being the set of natural numbers. Let * : A × A $\rightarrow$ A be defined as $(a, b) * (c, d) = \{ad + bc, bd)$ for all $(a, b), (c, d) \in$A. Show that

    (i) * is commutative

    (ii) * is associative

    (iii) identity element w.r.t * does not exist.

11. Let * be a binary operation on the set N of natural numbers defined by the rule $a * b = ab$ for all $a, b \in$N

    (i) Is * commutative? (ii) Is * associative?

ANSWERS

CHECK YOUR PROGRESS 23.2

1. (i) No (ii) Yes
2. (a), (b)
3. (a),
4. (a), (c),(e)
5. (a), (b)

CHECK YOUR PROGRESS 23.3

1. $\text{fog} = \dfrac{x^2}{(1-x)^2} + 2, \quad \text{gof} = \dfrac{x^2+2}{x^2+1}$

   $\text{fof} = x^4 + 4x^2 + 6, \quad \text{gog} = x$

2. (a) $\text{fog} = 4x^2 + 20x + 21, \ \text{gof} = 2x^2 - 3$

   $\text{fof} = x^4 - 8x^2 + 12, \ \text{gog} = 4x + 15$

   (b) $\text{fog} = 9, \ \text{gof} = 3, \ \text{fof} = x^4, \ \text{gog} = 3$

   (c) $\text{fog} = \dfrac{6-7x}{x}, \ \text{gof} = \dfrac{2}{3x-7}, \ \text{fof} = 9x - 28, \quad \text{gog} = x$

6. (a) $\text{fog} = \left|x^{\frac{1}{3}}\right|$ (b) $\text{goh} = \dfrac{1}{x^{\frac{1}{3}}}$ (c) $\text{foh} = \left|\dfrac{1}{x}\right|$

   (d) $\text{hog} = \dfrac{1}{x^{\frac{1}{3}}}$ (e) $\text{fogoh}(1) = 1$

CHECK YOUR PROGRESS 23.4

1. (ii) Domain is B. Range is A.

2. (a) $f^{-1}(x) = x - 3$ (b) $f^{-1}(x) = \dfrac{1-x}{3}$

   (c) Inverse does not exist. (d) $f^{-1}(x) = \dfrac{1}{x-1}$

CHECK YOUR PROGRESS 23.5

1. (i) No (ii) Yes (iii) Yes

2. 16

4. $\frac{9}{8}$

5. (0,0)

6. $\text{identity} = 0,\ a^{-1} = \frac{-a}{a+1}$

TERMINAL EXERCISE

1. (*a*) $fog = (4x-1)^3,\ gof = 4x^3 - 1,\ fog = x^9,\ gog = 16x - 5$

   (*b*) $fog = \frac{1}{(x^2 - 2x + 3)^2},\ gof = \frac{3x^4 - 2x^2 + 1}{x^4},\ fof = x^4, gog - x^4 - 4x^3 + 4x^2$

   (*c*) $fog = \sqrt{x-8},\ gof = \sqrt{x-4} - 4,\ fof = \sqrt{\sqrt{x-4-4}} gog = x - 8$

   (*d*) $fog = x^4 + 2x^2,\ gof = x^4 - 2x^2 + 2,\ fof = x^4 - 2x^2,\ gog = x^4 + 2x^2 + 2,$

2. (*a*) $\left|\frac{1}{x^{1/3}}\right|$, (*b*) $(fog)(3) = 364,\ (gof)(3) = 289$

3. (*c*), (*d*), (*e*),

4. $f(x) = \sin^2 x,\ g(x) = \sqrt{x}$

8. Neither reflexive, nor symmetric, nor transitive

9. Yes, R is an equivalence relation

11. (i) Not commutative

24

INVERSE TRIGONOMETRIC FUNCTIONS

In the previous lesson, you have studied the definition of a function and different kinds of functions. We have defined inverse function.

Let us briefly recall :

Let f be a one-one onto function from A to B.

Let y be an arbitary element of B. Then, f being onto, $\exists$ an element $x \in A$ such that $f(x) = y$. Also, f being one-one, then x must be unique. Thus for each $y \in B$, $\exists$ a unique element $x \in A$ such that $f(x) = y$. So we may define a function, denoted by f^{-1} as $f^{-1} : B \to A$

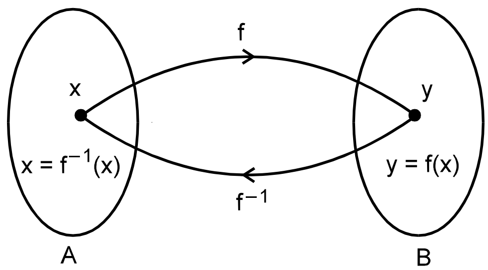


Fig. 24.1

$$\therefore \qquad f^{-1}(y) = x \Leftrightarrow f(x) = y$$

The above function f^{-1} is called the inverse of f. A function is invertiable if and only if f is one-one onto.

In this case the domain of f^{-1} is the range of f and the range of f^{-1} is the domain f.

Let us take another example.

We define a function : f : Car $\to$ Registration No.

If we write, g : Registration No. $\to$ Car, we see that the domain of f is range of g and the range of f is domain of g.

So, we say g is an **inverse function** of f, i.e., $g = f^{-1}$.

In this lesson, we will learn more about inverse trigonometric function, its domain and range, and simplify expressions involving inverse trigonometric functions.

OBJECTIVES

After studying this lesson, you will be able to :

- define inverse trigonometric functions;
- state the condition for the inverse of trigonometric functions to exist;
- define the principal value of inverse trigonometric functions;
- find domain and range of inverse trigonometric functions;
- state the properties of inverse trigonometric functions; and
- simplify expressions involving inverse trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of function and their types, domain and range of a function
- Formulae for trigonometric functions of sum, difference, multiple and sub-multiples of angles.

24.1 IS INVERSE OF EVERY FUNCTION POSSIBLE ?

Take two ordered pairs of a function (x_1, y) and (x_2, y)

If we invert them, we will get (y, x_1) and (y, x_2)

This is not a function because the first member of the two ordered pairs is the same.

Now let us take another function :

$$\left(\sin\frac{\pi}{2}, 1\right), \left(\sin\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\sin\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

Writing the inverse, we have

$$\left(1, \sin\frac{\pi}{2}\right), \left(\frac{1}{\sqrt{2}}, \sin\frac{\pi}{4}\right) \text{ and } \left(\frac{\sqrt{3}}{2}, \sin\frac{\pi}{3}\right)$$

which is a function.

Let us consider some examples from daily life.

f : Student $\rightarrow$ Score in Mathematics

Do you think f^{-1} will exist ?

It may or may not be because the moment two students have the same score, f^{-1} will cease to be a function. Because the first element in two or more ordered pairs will be the same. So we conclude that

every function is not invertible.

Example 24.1 If $f : \mathrm{R} \rightarrow \mathrm{R}$ defined by $f(x) = x^3 + 4$. What will be f^{-1}?

Solution : In this case f is one-to-one and onto both.

$\Rightarrow f$ is invertible.

Let $\qquad y = x^3 + 4$

$\therefore \qquad y - 4 = x^3 \Rightarrow x = \sqrt[3]{y-4}$

So f^{-1}, inverse function of f i.e., $f^{-1}(y) = \sqrt[3]{y-4}$

The functions that are one-to-one and onto will be invertible.

Let us extend this to trigonometry :

Take $y = \sin x$. Here domain is the set of all real numbers. Range is the set of all real numbers lying between -1 and 1, including -1 and 1 i.e. $-1 \le y \le 1$.

We know that there is a unique value of y for each given number x.

In inverse process we wish to know a number corresponding to a particular value of the sine.

Suppose $\quad y = \sin x = \frac{1}{2}$

$$\Rightarrow \qquad \sin x = \sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6} = ...$$

x may have the values as $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6} = ...$

Thus there are infinite number of values of x.

$y = \sin x$ can be represented as

$$\left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right), ...$$

The inverse relation will be

$$\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), ...$$

It is evident that it is not a function as first element of all the ordered pairs is $\frac{1}{2}$, which contradicts the definition of a function.

Consider $y = \sin x$, where $x \in R$ (domain) and $y \in [-1, 1]$ or $-1 \le y \le 1$ which is called range. This is many-to-one and onto function, therefore it is not invertible.

Can $y = \sin x$ be made invertible and how? Yes, if we restrict its domain in such a way that it becomes one-to-one and onto taking x as

(i)	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$,	$y \in [-1, 1]$	or
(ii)	$\frac{3\pi}{2} \le x \le \frac{5\pi}{2}$	$y \in [-1, 1]$	or
(iii)	$-\frac{5\pi}{2} \le x \le -\frac{3\pi}{2}$	$y \in [-1, 1]$	etc.

Now consider the inverse function $y = \sin^{-1} x$.

We know the domain and range of the function. We interchange domain and range for the inverse of the function. Therefore,

(i)	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$x \in [-1, 1]$	or
(ii)	$\frac{3\pi}{2} \le y \le \frac{5\pi}{2}$	$x \in [-1, 1]$	or

(iii) $-\frac{5\pi}{2} \le y \le -\frac{3\pi}{2}$ $x \in [-1, 1]$ etc.

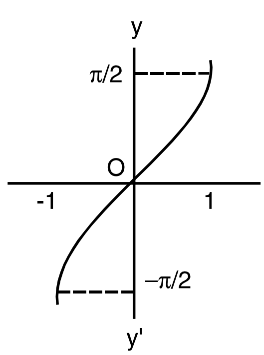
Here we take the least numerical value among all the values of the real number whose sine is x which is called the principle value of $\sin^{-1} x$.

For this the only case is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Therefore, for principal value of $y = \sin^{-1} x$, the domain is $[-1, 1]$ i.e. $x \in [-1, 1]$ and range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

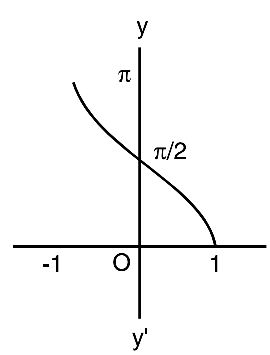
Similarly, we can discuss the other inverse trigonometric functions.

	Function	**Domain**	**Range (Principal value)**
1.	$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$	R	$[0, \pi]$
5.	$y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
6.	$y = \text{cosec}^{-1} x$	$x \ge 1$ or $x \le -1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

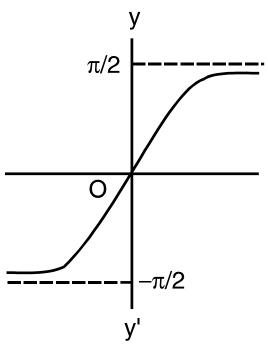
24.2 GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS



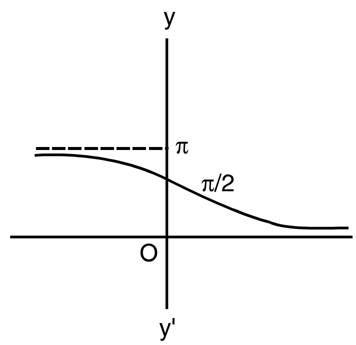
$y = \sin^{-1} x$



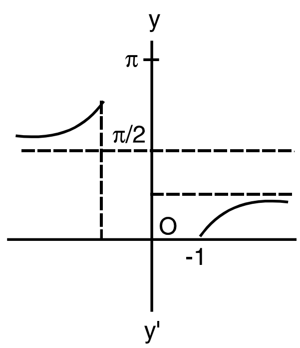
$y = \cos^{-1} x$



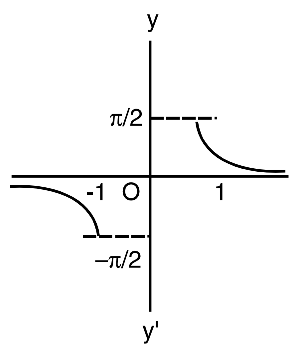
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{cosec}^{-1} x$

Fig. 24.2

Example 24.2 Find the principal value of each of the following :

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution : (i) Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

or $\sin\theta = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$ or $\theta = \frac{\pi}{4}$

(ii) Let $\cos^{-1}\left(-\frac{1}{2}\right) = \theta$

$\Rightarrow$ $\cos\theta = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$ or $\theta = \frac{2\pi}{3}$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \theta$ or $-\frac{1}{\sqrt{3}} = \tan\theta$ or $\tan\theta = \tan\left(-\frac{\pi}{6}\right)$

$\Rightarrow$ $\theta = -\frac{\pi}{6}$

Example 24.3 Find the principal value of each of the following :

(a) (i) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\tan^{-1}(-1)$

(b) Find the value of the following using the principal value :

$$\sec\left[\cos^{-1}\frac{\sqrt{3}}{2}\right]$$

Solution : (a) (i) Let $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\theta$, then

$$\frac{1}{\sqrt{2}}=\cos\theta \qquad \text{or} \qquad \cos\theta=\cos\frac{\pi}{4}$$

$$\Rightarrow \qquad \theta=\frac{\pi}{4}$$

(ii) Let $\tan^{-1}(-1)=\theta$, then

$$-1=\tan\theta \qquad \text{or} \qquad \tan\theta=\tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \qquad \theta=-\frac{\pi}{4}$$

(b) Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\theta$, then

$$\frac{\sqrt{3}}{2}=\cos\theta \qquad \text{or} \qquad \cos\theta=\cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \qquad \theta=\frac{\pi}{6}$$

$$\therefore \qquad \sec\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)=\sec\theta=\sec\left(\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}$$

Example 24.4 Simplify the following :

(i) $\cos\left(\sin^{-1}x\right)$ (ii) $\cot\left(\operatorname{cosec}^{-1}x\right)$

Solution : (i) Let $\sin^{-1}x=\theta$

$$\Rightarrow \qquad x=\sin\theta$$

$$\therefore \qquad \cos\left[\sin^{-1}x\right]=\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$$

(ii) Let $\operatorname{cosec}^{-1}x=\theta$

$$\Rightarrow \qquad x=\operatorname{cosec}\theta$$

Also $$\cot\theta = \sqrt{\operatorname{cosec}^2\theta - 1}$$
$$= \sqrt{x^2 - 1}$$

CHECK YOUR PROGRESS 24.1

1. Find the principal value of each of the following :

(a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (b) $\operatorname{cosec}^{-1}(-\sqrt{2})$ (c) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(d) $\tan^{-1}(-\sqrt{3})$ (e) $\cot^{-1}(1)$

2. Evaluate each of the following :

(a) $\cos\left(\cos^{-1}\frac{1}{3}\right)$ (b) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$ (c) $\cos\left(\operatorname{cosec}^{-1}\frac{2}{\sqrt{3}}\right)$

(d) $\tan\left(\sec^{-1}\sqrt{2}\right)$ (e) $\operatorname{cosec}\left[\cot^{-1}(-\sqrt{3})\right]$

3. Simplify each of the following expressions :

(a) $\sec\left(\tan^{-1}x\right)$ (b) $\tan\left(\operatorname{cosec}^{-1}\frac{x}{2}\right)$ (c) $\cot\left(\operatorname{cosec}^{-1}x^2\right)$

(d) $\cos\left(\cot^{-1}x^2\right)$ (e) $\tan\left(\sin^{-1}\left(\sqrt{1-x}\right)\right)$

24.3 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property 1 $\sin^{-1}(\sin\theta) = \theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Solution : Let $\sin\theta = x$

$\Rightarrow$ $$\theta = \sin^{-1}x$$
$$= \sin^{-1}(\sin\theta) = \theta$$

Also $\sin\left(\sin^{-1}x\right) = x$

Similarly, we can prove that

(i) $\cos^{-1}(\cos\theta) = \theta,\ 0 \le \theta \le \pi$

(ii) $\tan^{-1}(\tan\theta) = \theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Property 2 (i) $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$ (ii) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Solution : (i) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow \quad x = \operatorname{cosec}\theta$

$\Rightarrow \quad \frac{1}{x} = \sin\theta$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad \operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

(ii) Let $\cot^{-1} x = \theta$

$\Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \frac{1}{x} = \tan\theta$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\sec^{-1} x = \theta$

$\Rightarrow \quad x = \sec\theta$

$\therefore \quad \frac{1}{x} = \cos\theta \quad$ or $\quad \theta = \cos^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Property 3 (i) $\sin^{-1}(-x) = -\sin^{-1} x$ (ii) $\tan^{-1}(-x) = -\tan^{-1} x$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Solution : (i) Let $\sin^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \sin\theta \quad$ or $\quad x = -\sin\theta = \sin(-\theta)$

$\therefore \quad -\theta = \sin^{-1} x \quad$ or $\quad \theta = -\sin^{-1} x$

or $\quad \sin^{-1}(-x) = -\sin^{-1} x$

(ii) Let $\tan^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \tan\theta \quad$ or $\quad x = -\tan\theta = \tan(-\theta)$

$\therefore \quad \theta = -\tan^{-1} x \quad$ or $\quad \tan^{-1}(-x) = -\tan^{-1} x$

(iii) Let $\cos^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \cos\theta \quad$ or $\quad x = -\cos\theta = \cos(\pi - \theta)$

$\therefore \quad \cos^{-1} x = \pi - \theta$

$\therefore \quad \cos^{-1}(-x) = \pi - \cos^{-1} x$

Property 4. (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Soluton : (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Let $\sin^{-1} x = \theta \quad \Rightarrow \quad x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

or $\qquad \cos^{-1} x = \left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \qquad \theta + \cos^{-1} x = \frac{\pi}{2} \qquad$ or $\qquad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) Let $\cot^{-1} x = \theta \quad \Rightarrow \quad x = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$\therefore \qquad \tan^{-1} x = \frac{\pi}{2} - \theta \qquad$ or $\qquad \theta + \tan^{-1} x = \frac{\pi}{2}$

or $\qquad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$

(iii) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow \qquad$ ___ $x = \operatorname{cosec}\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

$\therefore \qquad \sec^{-1} x = \frac{\pi}{2} - \theta \qquad$ or $\qquad \theta + \sec^{-1} x = \frac{\pi}{2}$

$\Rightarrow \operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Property 5 (i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Solution : (i) Let $\tan^{-1} x = \theta$, $\tan^{-1} y = \phi \quad \Rightarrow x = \tan\theta, y = \tan\phi$

We have to prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

By substituting that above values on L.H.S. and R.H.S., we have

$$\text{L.H.S.} = \theta + \phi \text{ and } \text{R.H.S.} = \tan^{-1}\left[\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right]$$

$$= \tan^{-1}\left[\tan(\theta + \phi)\right] = \theta + \phi = \text{L.H.S.}$$

$\therefore$ The result holds.
Simiarly (ii) can be proved.

Property 6 $2\tan^{-1}x = \sin^{-1}\left[\frac{2x}{1+x^2}\right] = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right] = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

Let $x = \tan\theta$

Substituting in (i), (ii), (iii), and (iv) we get

$$2\tan^{-1}x = 2\tan^{-1}(\tan\theta) = 2\theta \qquad \text{......(i)}$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta \qquad \text{.....(ii)}$$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos^2\theta - \sin^2\theta)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta \qquad \text{......(iii)}$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) = 2\theta \qquad \text{.....(iv)}$$

From (i), (ii), (iii) and (iv), we get

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Property 7

(i) $$\sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$= \sec^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right] = \cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] = \operatorname{cosec}^{-1}\left[\frac{1}{x}\right]$$

(ii) $$\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$= \operatorname{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right] = \cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right] = \sec^{-1}\left[\frac{1}{x}\right]$$

Proof : Let $\sin^{-1}x = \theta \Rightarrow \sin\theta = x$

(i) $\cos\theta = \sqrt{1-x^2}$, $\tan\theta = \frac{x}{\sqrt{1-x^2}}$, $\sec\theta = \frac{1}{\sqrt{1-x^2}}$, $\cot\theta = \frac{\sqrt{1-x^2}}{x}$ and $\operatorname{cosec}\theta = \frac{1}{x}$

$$\therefore \qquad \sin^{-1}x = \theta = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) Let $\cos^{-1}x=\theta \Rightarrow \qquad x=\cos\theta$

$\therefore \qquad \sin\theta=\sqrt{1-x^2}, \quad \tan\theta=\dfrac{\sqrt{1-x^2}}{x}, \quad \sec\theta=\dfrac{1}{x}, \quad \cot\theta=\dfrac{x}{\sqrt{1-x^2}}$

and $\qquad \operatorname{cosec}\theta=\dfrac{1}{\sqrt{1-x^2}}$

$$\cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\sec^{-1}\left(\frac{1}{x}\right)$$

Example 24.5 Prove that

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$$

Solution : Applying the formula :

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ we have}$$

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right)=\tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\left(\frac{2}{9}\right)$$

Example 24.6 Prove that

$$\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

Solution : Let $\sqrt{x}=\tan\theta$ then

$$\text{L.H.S.} = \theta \text{ and R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$=\frac{1}{2}\times 2\theta=\theta$$

$\therefore \qquad$ L.H.S. = R.H.S.

Example 24.7 Solve the equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, \; x>0$$

Solution : Let $x = \tan\theta$, then

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] = \frac{1}{2}\theta, \Rightarrow \frac{\pi}{4}-\theta = \frac{1}{2}\theta, \Rightarrow \theta = \frac{\pi}{4}\times\frac{2}{3} = \frac{\pi}{6}$$

$$\therefore \quad x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Example 24.8 Show that

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Solution : Let $x^2 = \cos 2\theta$, then

$$2\theta = \cos^{-1}(x^2), \Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2$$

Substituting $x^2 = \cos 2\theta$ in L.H.S. of the given equation, we have

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

CHECK YOUR PROGRESS 24.2

1. Evaluate each of the following :

(a) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ (b) $\cot\left(\tan^{-1}\alpha+\cot^{-1}\alpha\right)$

(c) $\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)$

(d) $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ (e) $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$

2. If $\cos^{-1} x + \cos^{-1} y = \beta$, prove that $x^2 - 2xy\cos\beta + y^2 = \sin^2\beta$

3. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$

4. Prove each of the following :

(a) $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}} = \frac{\pi}{2}$ (b) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

(c) $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$ (d) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

5. Solve the equation $\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x)$

LET US SUM UP

- Inverse of a trigonometric function exists if we restrict the domain of it.

(i) $\sin^{-1} x = y$ if $\sin y = x$ where $-1 \le x \le 1, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii) $\cos^{-1} x = y$ if $\cos y = x$ where $-1 \le x \le 1, 0 \le y \le \pi$

(iii) $\tan^{-1} x = y$ if $\tan y = x$ where $x \in R, -\frac{\pi}{2} < y < \frac{\pi}{2}$

(iv) $\cot^{-1} x = y$ if $\cot y = x$ where $x \in R, 0 < y < \pi$

(v) $\sec^{-1} x = y$ if $\sec y = x$ where $x \ge 1,\ 0 \le y < \frac{\pi}{2}$ or $x \le -1, \frac{\pi}{2} < y \le \pi$

(vi) $\operatorname{cosec}^{-1} x = y$ if $\operatorname{cosec} y = x$ where $x \ge 1, 0 < y \le \frac{\pi}{2}$

or $x \le -1, -\frac{\pi}{2} \le y < 0$

- Graphs of inverse trigonometric functions can be represented in the given intervals by interchanging the axes as in case of $y = \sin x$, etc.

- **Properties :**

(i) $\sin^{-1}(\sin\theta) = \theta$, $\tan^{-1}(\tan\theta) = \theta$, $\tan(\tan^{-1}\theta) = \theta$ and $\sin(\sin^{-1}\theta) = \theta$

(ii) $\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$, $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$, $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

(iii) $\sin^{-1}(-x) = -\sin^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(iv) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

(v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(vi) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $\sin^{-1} x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Inverse_trigonometric_functions

http://mathworld.wolfram.com/InverseTrigonometricFunctions.html

TERMINAL EXERCISE

1. Prove each of the following :

(a) $\sin^{-1}\left(\dfrac{3}{5}\right) + \sin^{-1}\left(\dfrac{8}{17}\right) = \sin^{-1}\left(\dfrac{77}{85}\right)$

(b) $\tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{1}{9}\right) = \dfrac{1}{2}\cos^{-1}\left(\dfrac{3}{5}\right)$

(c) $\cos^{-1}\left(\dfrac{4}{5}\right) + \tan^{-1}\left(\dfrac{3}{5}\right) = \tan^{-1}\left(\dfrac{27}{11}\right)$

2. Prove each of the following :

(a) $2\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) = \tan^{-1}\left(\dfrac{23}{11}\right)$

(b) $\tan^{-1}\left(\dfrac{1}{2}\right) + 2\tan^{-1}\left(\dfrac{1}{3}\right) = \tan^{-1} 2$

(c) $\tan^{-1}\left(\dfrac{1}{8}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) = \tan^{-1}\left(\dfrac{1}{3}\right)$

3. (a) Prove that $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(b) Prove that $2\cos^{-1} x = \cos^{-1}\left(2x^2 - 1\right)$

(c) Prove that $\cos^{-1} x = 2\sin^{-1}\left(\sqrt{\dfrac{1-x}{2}}\right) = 2\cos^{-1}\left(\sqrt{\dfrac{1+x}{2}}\right)$

4. Prove the following :

(a) $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)=\dfrac{\pi}{4}-\dfrac{x}{2}$

(b) $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)=\dfrac{\pi}{4}-x$

(c) $\cot^{-1}\left(\dfrac{ab+1}{a-b}\right)+\cot^{-1}\left(\dfrac{bc+1}{b-c}\right)+\cot^{-1}\left(\dfrac{ca+1}{c-a}\right)=0$

5. Solve each of the following :

(a) $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}$

(b) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

(c) $\cos^{-1}x+\sin^{-1}\left(\dfrac{1}{2}x\right)=\dfrac{\pi}{6}$

(d) $\cot^{-1}x-\cot^{-1}(x+2)=\dfrac{\pi}{12},\ x>0$

ANSWERS

CHECK YOUR PROGRESS 24.1

1. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{\pi}{3}$ (e) $\frac{\pi}{4}$

2. (a) $\frac{1}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{1}{2}$ (d) 1 (e) -2

3. (a) $\sqrt{1+x^2}$ (b) $\frac{2}{\sqrt{x^2-4}}$ (c) $\sqrt{x^4-1}$ (d) $\frac{x^2}{\sqrt{x^4+1}}$ (e) $\sqrt{\frac{1-x}{x}}$

CHECK YOUR PROGRESS 24.2

1. (a) 1 (b) 0 (c) $\frac{x+y}{1-xy}$ (d) $\frac{5}{12}$ (e) $-\frac{7}{17}$

5. $0, \pm\frac{1}{2}$

TERMINAL EXERCISE

5. (a) $\frac{1}{6}$ (b) $\frac{\pi}{4}$ (c) ± 1 (d) $\sqrt{3}$

25

LIMIT AND CONTINUITY

Consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$

You can see that the function f(x) is not defined at x = 1 as $x - 1$ is in the denominator. Take the value of x very nearly equal to but not equal to 1 as given in the tables below. In this case $x - 1 \neq 0$ as $x \neq 1$.

$\therefore$ We can write $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{(x-1)} = x + 1$, because $x - 1 \neq 0$ and so division by $(x - 1)$ is possible.

Table -1

x	f(x)
0.5	1.5
0.6	1.6
0.7	1.7
0.8	1.8
0.9	1.9
0.91	1.91
:	:
:	:
0.99	1.99
:	:
:	:
0.9999	1.9999

Table - 2

x	f (x)
1.9	2.9
1.8	2.8
1.7	2.7
1.6	2.6
1.5	2.5
:	:
:	:
1.1	2.1
1.01	2.01
1.001	2.001
:	:
:	:
1.00001	2.00001

In the above tables, you can see that as x gets closer to 1, the corresponding value of f (x) also gets closer to 2.

However, in this case f(x) is not defined at x = 1. The idea can be expressed by saying that the limiting value of f(x) is 2 when x approaches to 1.

Let us consider another function f (x) =2x. Here, we are interested to see its behavior near the point 1 and at x = 1. We find that as x gets nearer to 1, the corresponding value of f (x) gets closer to 2 at x = 1 and the value of f (x) is also 2.

So from the above findings, what more can we say about the behaviour of the function near x = 2 and at x = 2 ?

In this lesson we propose to study the behaviour of a function near and at a particular point where the function may or may not be defined.

OBJECTIVES

After studying this lesson, you will be able to :

- define limit of a function
- derive standard limits of a function
- evaluate limit using different methods and standard limits.
- define and interprete geometrically the continuity of a function at a point;
- define the continuity of a function in an interval;
- determine the continuity or otherwise of a function at a point; and
- state and use the theorems on continuity of functions with the help of examples.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a function
- Drawing the graph of a function
- Concept of trigonometric function
- Concepts of exponential and logarithmic functions

25.1 LIMIT OF A FUNCTION

In the introduction, we considered the function $f(x)=\frac{x^2-1}{x-1}$. We have seen that as x approaches l, f (x) approaches 2. In general, if a function f (x) approaches L when x approaches 'a', we say that L is the limiting value of f (x)

Symbolically it is written as

$$\lim_{x\to a} f\left(x\right)=L$$

Now let us find the limiting value of the function $\left(5x-3\right)$ when x approaches 0.

i.e. $$\lim_{x\to 0}\left(5x-3\right)$$

For finding this limit, we assign values to x from left and also from right of 0.

x	−0.1	−0.01	−0.001	−0.0001..........
5x − 3	−3.5	−3.05	−3.005	−3.0005..........

x	0.1	0.01	0.001	0.0001..........
5x − 3	−2.5	−2.95	−2.995	−2.9995.........

It is clear from the above that the limit of $(5x-3)$ as $x \to 0$ is -3

i.e., $$\lim_{x\to 0}(5x-3) = -3$$

This is illustrated graphically in the Fig. 20.1

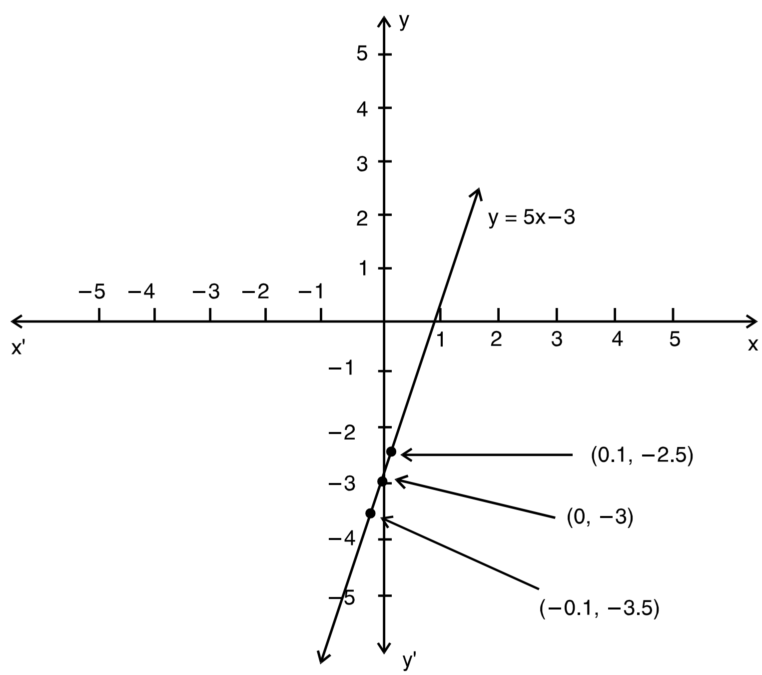


Fig. 25.1

The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as x (independent variable) ends to a finite quantity, say a

Consider an example : Find $\lim\limits_{x\to 3} f(x)$, where $f(x) = \dfrac{x^2-9}{x-3}$

We can solve it by the method of substitution. Steps of which are as follows :

Remarks : It may be noted that f (3) is not defined, however, in this case the limit of the

Step 1: We consider a value of x close to a say $x = a + h$, where h is a very small positive number. Clearly, as $x \to a$, $h \to 0$	For $f(x) = \frac{x^2 - 9}{x - 3}$ we write $x = 3 + h$, so that as $x \to 3, h \to 0$
Step 2 : Simplify $f(x) = f(a+h)$	Now $f(x) = f(3+h)$ $= \frac{(3+h)^2 - 9}{3 + h - 3}$ $= \frac{h^2 + 6h}{h}$ $= h + 6$
Step 3 : Put $h = 0$ and get the requried result	$\therefore \lim_{x\to 3} f(x) = \lim_{h\to 0}(6+h)$ As $x \to 0$, $h \to 0$ Thus, $\lim_{x\to 3} f(x) = 6 + 0 = 6$ by putting $h = 0$.

function $f(x)$ as $x \to 3$ is 6.

Now we shall discuss other methods of finding limits of different types of functions.

Consider the example :

Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} \frac{x^3 - 1}{x^2 - 1}, & x \neq 1 \\ 1 & , x = 1 \end{cases}$

Here, for $x \neq 1$, $f(x) = \frac{x^3 - 1}{x^2 - 1} = \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)}$

It shows that if f (x) is of the form $\frac{g(x)}{h(x)}$, then we may be able to solve it by the method of factors. In such case, we follow the following steps :

Step 1. Factorise g (x) and h (x)	**Sol.** $f(x)=\dfrac{x^3-1}{x^2-1}$ $=\dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$ ($\because x\neq 1, \therefore x-1\neq 0$ and as such can be cancelled)
Step 2 : Simplify f (x)	$\therefore \quad f(x)=\dfrac{x^2+x+1}{x+1}$
Step 3 : Putting the value of x, we get the required limit.	$\therefore \lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}=\dfrac{1+1+1}{1+1}=\dfrac{3}{2}$ Also $f(1)=1$(given) In this case, $\lim\limits_{x\to 1} f(x)\neq f(1)$

Thus, the limit of a function f (x) as $x\to a$ may be different from the value of the functioin at $x=a$.

Now, we take an example which cannot be solved by the method of substitutions or method of factors.

Evaluate $\quad \lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Here, we do the following steps :

Step 1. Rationalise the factor containing square root.

Step 2. Simplify.

Step 3. Put the value of x and get the required result.

Solution :

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{\sqrt{(1+x)^2}-\sqrt{(1-x)^2}}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

Notes

$$= \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x \to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = 1$$

25.2 LEFT AND RIGHT HAND LIMITS

You have already seen that $x \to a$ means x takes values which are very close to 'a', i.e. either the value is greater than 'a' or less than 'a'.

In case x takes only those values which are less than 'a' and very close to 'a' then we say x is approaches 'a' from the left and we write it as $x \to a^{-}$. Similarly, if x takes values which are greater than 'a' and very close to 'a' then we say x is approaching 'a' from the right and we write it as $x \to a^{+}$.

Thus, if a function f (x) approaches a limit ℓ_1, as x approaches 'a' from left, we say that the left hand limit of $f(x)$ as $x \to a$ is ℓ_1.

We denote it by writing

$$\lim_{x \to a^{-}} f(x) = \ell_1 \qquad \text{or} \qquad \lim_{h \to 0} f(a-h) = \ell_1, \, h > 0$$

Similarly, if f (x) approaches the limit ℓ_2, as x approaches 'a' from right we say, that the right hand limit of $f(x)$ as $x \to a$ is ℓ_2.

We denote it by writing

$$\lim_{x \to a^{+}} f(x) = \ell_2 \qquad \text{or} \qquad \lim_{h \to 0} f(a+h) = \ell_2, \, h > 0$$

Working Rules

Finding the right hand limit i.e.,

$$\lim_{x \to a^{+}} f(x)$$

Put $x = a + h$

Find $\lim_{h \to 0} f(a+h)$

Finding the left hand limit, i.e,

$$\lim_{x \to a^{-}} f(x)$$

Put $x = a - h$

Find $\lim_{h \to 0} f(a-h)$

Note : In both cases remember that h takes only positive values.

25.3 LIMIT OF FUNCTION y = f(x) AT x = a

Consider an example :

Find $\lim_{x\to1} f(x)$, where $f(x) = x^2 + 5x + 3$

Here $$\lim_{x\to1^+} f(x) = \lim_{h\to0}\left[(1+h)^2 + 5(1+h) + 3\right]$$

$$= \lim_{h\to0}\left[1 + 2h + h^2 + 5 + 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad \text{.....(i)}$$

and $$\lim_{x\to1^-} f(x) = \lim_{h\to0}\left[(1-h)^2 + 5(1-h) + 3\right]$$

$$= \lim_{x\to0}\left[1 - 2h + h^2 + 5 - 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad \text{.....(ii)}$$

From (i) and (ii), $\lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x)$

Now consider another example :

Evaluate : $$\lim_{x\to3} \frac{|x-3|}{x-3}$$

Here $$\lim_{x\to3^+} \frac{|x-3|}{x-3} = \lim_{h\to0} \frac{|(3+h)-3|}{[(3+h)-3]}$$

$$= \lim_{h\to0} \frac{|h|}{h} = \lim_{h\to0} \frac{h}{h} \text{ (as } h>0\text{, so } |h| = h)$$

$$= 1 \qquad \text{.....(iii)}$$

and $$\lim_{x\to3^-} \frac{|x-3|}{x-3} = \lim_{h\to0} \frac{|(3-h)-3|}{[(3-h)-3]}$$

$$= \lim_{h\to0} \frac{|-h|}{-h} = \lim_{h\to0} \frac{h}{-h} \quad \text{(as } h > 0\text{, so } |-h| = h)$$

$$= -1 \qquad \text{.....(iv)}$$

$\therefore$ From (iii) and (iv), $\lim_{x\to3^+} \frac{|x-3|}{x-3} \neq \lim_{x\to3^-} \frac{|x-3|}{x-3}$

Thus, in the first example right hand limit = left hand limit whereas in the second example right hand limit $\neq$ left hand limit.

Hence the left hand and the right hand limits may not always be equal.

We may conclude that

$\lim\limits_{x \to 1} \left(x^2 + 5x + 3\right)$ exists (which is equal to 9) and $\lim\limits_{x \to 3} \dfrac{|x-3|}{x-3}$ does not exist.

Note :

I $\quad \left.\begin{array}{l} \lim\limits_{x \to a^+} f(x) = \ell \\ \text{and} \quad \lim\limits_{x \to a^-} f(x) = \ell \end{array}\right\} \Rightarrow \lim\limits_{x \to a} f(x) = \ell$

II $\quad \left.\begin{array}{l} \lim\limits_{x \to a^+} f(x) = \ell_1 \\ \text{and} \quad \lim\limits_{x \to a^-} f(x) = \ell_2 \end{array}\right\} \Rightarrow \lim\limits_{x \to a} f(x)$ does not exist.

III $\quad \lim\limits_{x \to a^+} f(x)$ or $\lim\limits_{x \to a^-} f(x)$ does not exist $\Rightarrow \lim\limits_{x \to a} f(x)$ does not exist.

25.4 BASIC THEOREMS ON LIMITS

1. $\lim\limits_{x \to a} cx = c \lim\limits_{x \to a} x$, c being a constant.

To verify this, consider the function $f(x) = 5x$.

We observe that in $\lim\limits_{x \to 2} 5x$, 5 being a constant is not affected by the limit.

$$\therefore \quad \lim_{x \to 2} 5x = 5 \lim_{x \to 2} x$$

$$= 5 \times 2 = 10$$

2. $$\lim_{x \to a} \left[g(x) + h(x) + p(x) + \ldots\right] = \lim_{x \to a} g(x) + \lim_{x \to a} h(x) + \lim_{x \to a} p(x) + \ldots\ldots$$

where $g(x), h(x), p(x), \ldots$ are any function.

3. $$\lim_{x \to a} \left[f(x) \cdot g(x)\right] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

To verify this, consider $f(x) = 5x^2 + 2x + 3$

and $g(x) = x + 2$.

Then $$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(5x^2 + 2x + 3\right)$$

$$= 5 \lim_{x \to 0} x^2 + 2 \lim_{x \to 0} x + 3 = 3$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(x+2) = \lim_{x\to 0} x+2 = 2$$

$\therefore$
$$\lim_{x\to 0}(5x^2+2x+3)\lim_{x\to 0}(x+2) = 6 \qquad .....(i)$$

Again
$$\lim_{x\to 0}[f(x)\cdot g(x)] = \lim_{x\to 0}[(5x^2+2x+3)(x+2)]$$

$$= \lim_{x\to 0}(5x^3+12x^2+7x+6)$$

$$= 5\lim_{x\to 0} x^3 + 12\lim_{x\to 0} x^2 + 7\lim_{x\to 0} x + 6$$

$$= 6 \qquad .....(ii)$$

From (i) and (ii), $\lim_{x\to 0}[(5x^2+2x+3)(x+2)] = \lim_{x\to 0}(5x^2+2x+3)\lim_{x\to 0}(x+2)$

4. $\lim_{x\to a}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$

To verify this, consider the function $f(x) = \dfrac{x^2+5x+6}{x+2}$

we have
$$\lim_{x\to -1}(x^2+5x+6) = (-1)^2+5(-1)+6 = 1-5+6 = 2$$

and
$$\lim_{x\to -1}(x+2) = -1+2 = 1$$

$$\frac{\lim_{x\to -1}(x^2+5x+6)}{\lim_{x\to -1}(x+2)} = \frac{2}{1} = 2 \qquad .....(i)$$

Also
$$\lim_{x\to -1}\frac{(x^2+5x+6)}{x+2} = \lim_{x\to -1}\frac{(x+3)(x+2)}{x+2} \qquad \left[\begin{array}{l}\because x^2+5x+6\\ = x^2+3x+2x+6\\ = x(x+3)+2(x+3)\\ = (x+3)(x+2)\end{array}\right]$$

$$= \lim_{x\to -1}(x+3)$$

$$= -1+3 = 2 \qquad .....(ii)$$

$\therefore$ From (i) and (ii),

$$\lim_{x\to -1}\frac{x^2+5x+6}{x+2}=\frac{\lim\limits_{x\to -1}\left(x^2+5x+6\right)}{\lim\limits_{x\to -1}(x+2)}$$

We have seen above that there are many ways that two given functions may be combined to form a new function. The limit of the combined function as $x\to a$ can be calculated from the limits of the given functions. To sum up, we state below some basic results on limits, which can be used to find the limit of the functions combined with basic operations.

If $\lim\limits_{x\to a} f(x)=\ell$ and $\lim\limits_{x\to a} g(x)=m$, then

(i) $\lim\limits_{x\to a} kf(x)=k\lim\limits_{x\to a} f(x)=k\ell$ where k is a constant.

(ii) $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)=\ell\pm m$

(iii) $\lim\limits_{x\to a}\left[f(x)\cdot g(x)\right]=\lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)=\ell\cdot m$

(iv) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\dfrac{\ell}{m}$, provided $\lim\limits_{x\to a} g(x)\neq 0$

The above results can be easily extended in case of more than two functions.

Example 25.1 Find $\lim\limits_{x\to 1} f(x)$, where

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 1, & x=1\end{cases}$$

Solution : $f(x)=\dfrac{x^2-1}{x-1}=\dfrac{(x-1)(x+1)}{x-1}=(x+1)$ $[\because x\neq 1]$

$\therefore$ $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}(x+1)=1+1=2$

Note : $\dfrac{x^2-1}{x-1}$ is not defined at x=1. The value of $\lim\limits_{x\to 1} f(x)$ is independent of the value of $f(x)$ at x = 1.

Example 25.2 Evaluate : $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$.

Solution : $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)} = \lim_{x \to 2} \left(x^2+2x+4\right) \qquad [\because x \neq 2]$$

$$= 2^2 + 2 \times 2 + 4 = 12$$

Example 25.3 Evaluate : $\lim\limits_{x \to 2} \dfrac{\sqrt{3-x}-1}{2-x}$.

Solution : Rationalizing the numerator, we have

$$\frac{\sqrt{3-x}-1}{2-x} = \frac{\sqrt{3-x}-1}{2-x} \times \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} = \frac{3-x-1}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$\therefore \quad \lim_{x \to 2} \frac{\sqrt{3-x}-1}{2-x} = \lim_{x \to 2} \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \lim_{x \to 2} \frac{1}{\left(\sqrt{3-x}+1\right)} = \frac{1}{\left(\sqrt{3-2}+1\right)} = \frac{1}{1+1} = \frac{1}{2}$$

Example 25.4 Evaluate : $\lim\limits_{x \to 3} \dfrac{\sqrt{12-x}-x}{\sqrt{6+x}-3}$.

Solution : Rationalizing the numerator as well as the denominator, we get

$$\lim_{x \to 3} \frac{\sqrt{12-x}-x}{\sqrt{6+x}-3} = \lim_{x \to 3} \frac{\left(\sqrt{12-x}-x\right)\left(\sqrt{12-x}+x\right) \cdot \left(\sqrt{6+x}+3\right)}{\sqrt{6+x}-3\left(\sqrt{6+x}+3\right)\left(\sqrt{12-x}+x\right)}$$

$$= \lim_{x \to 3} \frac{\left(12-x-x^2\right)}{6+x-9} \cdot \lim_{x \to 3} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x}$$

$$= \lim_{x \to 3} \frac{-(x+4)(x-3)}{(x-3)} \cdot \lim_{x \to 3} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x} \qquad [\because x \neq 3]$$

$$= -(3+4) \cdot \frac{6}{6} = -7$$

Note : Whenever in a function, the limits of both numerator and denominator are zero, you should simplify it in such a manner that the denominator of the resulting function is not zero.

However, if the limit of the denominator is 0 and the limit of the numerator is non zero, then the limit of the function does not exist.

Let us consider the example given below :

Example 25.5 Find $\lim_{x \to 0} \frac{1}{x}$, if it exists.

Solution : We choose values of x that approach 0 from both the sides and tabulate the correspondling values of $\frac{1}{x}$.

x	-0.1	$-.01$	$-.001$	$-.0001$
$\frac{1}{x}$	-10	-100	-1000	-10000

x	0.1	$.01$	$.001$	$.0001$
$\frac{1}{x}$	10	100	1000	10000

We see that as $x \to 0$, the corresponding values of $\frac{1}{x}$ are not getting close to any number.

Hence, $\lim_{x \to 0} \frac{1}{x}$ does not exist. This is illustrated by the graph in Fig. 20.2

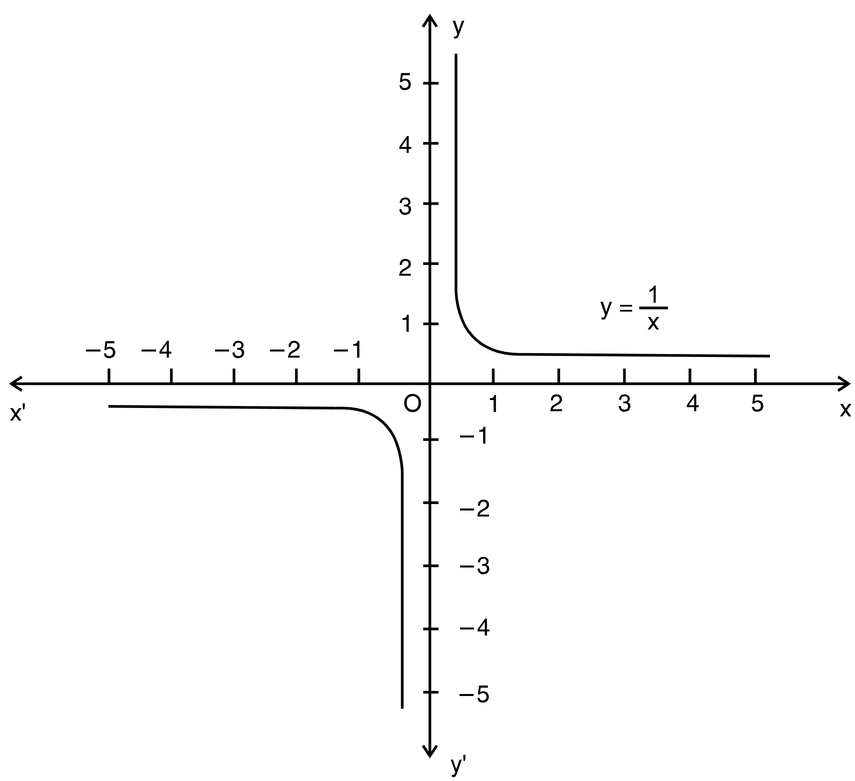


Fig. 25.2

Example 25.6 Evaluate : $\lim_{x \to 0} (|x| + |-x|)$

Solution : Since |x| has different values for $x \geq 0$ and x<0, therefore we have to find out both left hand and right hand limits.

$$\lim_{x \to 0^-} (|x| + |-x|) = \lim_{h \to 0} (|0-h| + |-(0-h)|)$$

$$= \lim_{h \to 0} (|-h| + |-(-h)|)$$

$$= \lim_{h \to 0} h + h = \lim_{h \to 0} 2h = 0 \qquad \text{...(i)}$$

and $$\lim_{x \to 0^+} (|x| + |-x|) = \lim_{h \to 0} (|0+h| + |-(0+h)|)$$

$$= \lim_{x \to 0} h + h = \lim_{h \to 0} 2h = 0 \qquad \text{...(ii)}$$

From (i) and (ii),

$$\lim_{x \to 0^-} (|x| + |-x|) = \lim_{h \to 0^+} [|x| + |-x|]$$

Thus, $$\lim_{h \to 0} [|x| + |-x|] = 0$$

Note : We should remember that left hand and right hand limits are specially used when (a) the functions under consideration involve modulus function, and (b) function is defined by more than one rule.

Example 25.7 Find the vlaue of 'a' so that

$\lim_{x \to 1} f(x)$ exist, where $f(x) = \begin{cases} 3x+5, & x \leq 1 \\ 2x+a, & x > 1 \end{cases}$

Solution :

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3x+5) \qquad [\because f(x) = 3x+5 \text{ for } x \leq 1]$$

$$= \lim_{h \to 0} [3(1-h)+5]$$

$$= 3 + 5 = 8 \qquad \text{.....(i)}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2x+a) \qquad [\because f(x) = 2x+a \text{ for } x > 1]$$

$$= \lim_{h \to 0} (2(1+h)+a)$$

$$= 2 + a \qquad \text{.....(ii)}$$

We are given that $\lim_{x \to 1} f(x)$ will exists provided

$$\Rightarrow \quad \lim_{x \to 1^-} = \lim_{x \to 1^+} f(x)$$

$\therefore$ From (i) and (ii),

$$2 + a = 8$$

$\therefore$ or, $a = 6$

Example 25.8 If a function f (x) is defined as

$$f(x)=\begin{cases} x & , \quad 0 \le x < \frac{1}{2} \\ 0 & , \quad x=\frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 \end{cases}$$

Examine the existence of $\lim\limits_{x \to \frac{1}{2}} f(x)$.

Solution : Here

$$f(x)=\begin{cases} x & , \quad 0 \le x < \frac{1}{2} & \text{.....(i)} \\ 0 & , \quad x=\frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 & \text{.....(ii)} \end{cases}$$

$$\lim_{x \to \left(\frac{1}{2}\right)^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2}-h\right)$$

$$= \lim_{x \to 0}\left(\frac{1}{2}-h\right) \quad \left[\because \frac{1}{2}-h<\frac{1}{2} \text{ and from (i), } f\left(\frac{1}{2}-h\right)=\frac{1}{2}-h\right]$$

$$=\frac{1}{2}-0=\frac{1}{2} \qquad \text{.....(iii)}$$

$$\lim_{x \to \left(\frac{1}{2}\right)^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2}+h\right)$$

$$= \lim_{h \to 0}\left[\left(\frac{1}{2}+h\right)-1\right] \left[\because \frac{1}{2}+h>\frac{1}{2} \text{ and from (ii), } f\left(\frac{1}{2}+h\right)=\left(\frac{1}{2}+h\right)-1\right]$$

$$=\frac{1}{2}+-1$$

$$=-\frac{1}{2} \qquad \text{.....(iv)}$$

From (iii) and (iv), left hand limit $\neq$ right hand limit

$\therefore \quad \lim\limits_{x \to \frac{1}{2}} f(x)$ does not exist.

CHECK YOUR PROGRESS 25.1

1. Evaluate each of the following limits :

(a) $\lim\limits_{x\to 2}[2(x+3)+7]$ (b) $\lim\limits_{x\to 0}\left(x^2+3x+7\right)$ (c) $\lim\limits_{x\to 1}\left[(x+3)^2-16\right]$

(d) $\lim\limits_{x\to -1}\left[(x+1)^2+2\right]$ (e) $\lim\limits_{x\to 0}\left[(2x+1)^3-5\right]$ (f) $\lim\limits_{x\to 1}(3x+1)(x+1)$

2. Find the limits of each of the following functions :

(a) $\lim\limits_{x\to 5}\dfrac{x-5}{x+2}$ (b) $\lim\limits_{x\to 1}\dfrac{x+2}{x+1}$ (c) $\lim\limits_{x\to -1}\dfrac{3x+5}{x-10}$

(d) $\lim\limits_{x\to 0}\dfrac{px+q}{ax+b}$ (e) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$ (f) $\lim\limits_{x\to -5}\dfrac{x^2-25}{x+5}$

(g) $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{x^2-3x+2}$ (h) $\lim\limits_{x\to \frac{1}{3}}\dfrac{9x^2-1}{3x-1}$

3. Evaluate each of the following limits:

(a) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$ (b) $\lim\limits_{x\to 0}\dfrac{x^3+7x}{x^2+2x}$ (c) $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}$

(d) $\lim\limits_{x\to 1}\left[\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right]$

4. Evaluate each of the following limits :

(a) $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-\sqrt{4-x}}{x}$ (b) $\lim\limits_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (c) $\lim\limits_{x\to 3}\dfrac{\sqrt{3+x}-\sqrt{6}}{x-3}$

(d) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-1}$ (e) $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-x}{2-\sqrt{6-x}}$

5. (a) Find $\lim\limits_{x\to 0}\dfrac{2}{x}$, if it exists. (b) Find $\lim\limits_{x\to 2}\dfrac{1}{x-2}$, if it exists.

6. Find the values of the limits given below :

(a) $\lim\limits_{x\to 0}\dfrac{x}{5-|x|}$ (b) $\lim\limits_{x\to 2}\dfrac{1}{|x+2|}$ (c) $\lim\limits_{x\to 2}\dfrac{1}{|x-2|}$

(d) Show that $\lim\limits_{x\to 5}\dfrac{|x-5|}{x-5}$ does not exist.

7. (a) Find the left hand and right hand limits of the function

$$f(x)=\begin{cases}-2x+3, & x\le 1\\ 3x-5, & x>1\end{cases} \text{ as } x\to 1$$

(b) If $f(x)=\begin{cases}x^2, & x\le 1\\ 1, & x>1\end{cases}$, find $\lim\limits_{x\to 1} f(x)$

(c) Find $\lim_{x \to 4} f(x)$ if it exists, given that $f(x) = \begin{cases} 4x+3, & x<4 \\ 3x+7, & x \geq 4 \end{cases}$

8. Find the value of 'a' such that $\lim_{x \to 2} f(x)$ exists,where $f(x) = \begin{cases} ax+5, & x<2 \\ x-1, & x \geq 2 \end{cases}$

9. Let $f(x) = \begin{cases} x, & x<1 \\ 1, & x=1 \\ x^2, & x>1 \end{cases}$

Establish the existence of $\lim_{x \to 1} f(x)$.

10. Find $\lim_{x \to 2} f(x)$ if it exists, where

$$f(x) = \begin{cases} x-1, & x<2 \\ 1, & x=2 \\ x+1, & x>2 \end{cases}$$

25.5 FINDING LIMITS OF SOME OF THE IMPORTANT FUNCTIONS

(i) Prove that $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ where n is a positive integer.

Proof : $\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{h \to 0} \frac{(a+h)^n - a^n}{a+h-a}$

$$= \lim_{h \to 0} \frac{\left(a^n + n\,a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots + h^n\right) - a^n}{h}$$

$$= \lim_{h \to 0} \frac{h\left(n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \ldots + h^{n-1}\right)}{h}$$

$$= \lim_{h \to 0} \left[n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \ldots + h^{n-1}\right]$$

$$= n\,a^{n-1} + 0 + 0 + \ldots + 0$$

$$= n\,a^{n-1}$$

$$\therefore \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$$

Note : However, the result is true for all n

(ii) Prove that (a) $\lim\limits_{x \to 0} \sin x = 0$ **and (b)** $\lim\limits_{x \to 0} \cos x = 1$

Proof : Consider a unit circle with centre B, in which $\angle C$ is a right angle and $\angle ABC = x$ radians.

Now sin x = A C and cos x = BC

As x decreases, A goes on coming nearer and nearer to C.

i.e., when $x \to 0, A \to C$

or when $x \to 0, AC \to 0$

and $BC \to AB, \text{i.e.,} BC \to 1$

$\therefore$ When $x \to 0$ $\sin x \to 0$ and $\cos x \to 1$

Thus we have

$$\lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos x = 1$$

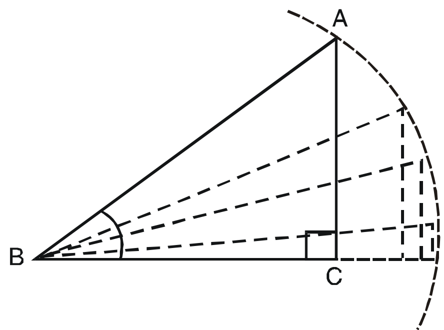


Fig. 25.3

(iii) Prove that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

Proof : Draw a circle of radius 1 unit and with centre at the origin *O*. Let B (1,0) be a point on the circle. Let A be any other point on the circle. Draw $AC \perp OX$.

Let $\angle AOX = x$ radians, where $0<x<\dfrac{\pi}{2}$

Draw a tangent to the circle at B meeting OA produced at D. Then $BD \perp OX$.

Area of $\Delta AOC <$ area of sector $OBA <$ area of ΔOBD.

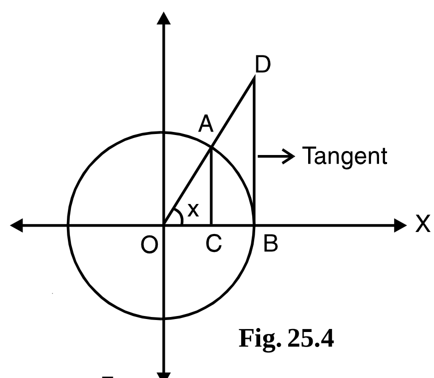


Fig. 25.4

$$\text{or} \frac{1}{2} OC \times AC < \frac{1}{2} x(1)^2 < \frac{1}{2} OB \times BD$$

$$\left[\because \text{area of triangle} = \frac{1}{2} \text{base} \times \text{height and area of sector} = \frac{1}{2} \theta\, r^2\right]$$

$$\therefore \qquad \frac{1}{2}\cos x \sin x < \frac{1}{2} x < \frac{1}{2} \cdot 1 \cdot \tan x$$

$$\left[\because \cos x = \frac{OC}{OA}, \sin x = \frac{AC}{OA} \text{ and } \tan x = \frac{BD}{OB}, OA = 1 = OB\right]$$

i.e., $\quad \cos x < \dfrac{x}{\sin x} < \dfrac{\tan x}{\sin x}$ [Dividing throughout by $\dfrac{1}{2}\sin x$]

or $$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or $$\frac{1}{\cos x} > \frac{\sin x}{x} < \cos x$$

i.e., $$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$$

Taking limit as $x \to 0$, we get

$$\lim_{x\to 0} \cos x < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} \frac{1}{\cos x}$$

or $$1 < \lim_{x\to 0} \frac{\sin x}{x} < 1 \qquad \left[\because \lim_{x\to 0} \cos x = 1 \text{ and } \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1\right]$$

Thus, $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Note : In the above results, it should be kept in mind that the angle x must be expressed in radians.

(iv) Prove that $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

Proof : By Binomial theorem, when $|x| < 1$, we get

$$(1+x)^{\frac{1}{x}} = \left[1 + \frac{1}{x}\cdot x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2 + + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3 + \text{..........}\infty\right]$$

$$= \left[1 + 1 + \frac{(1-x)}{2!} + \frac{(1-x)(1-2x)}{3!} + \text{..........}\infty\right]$$

$$\therefore \lim_{x\to 0} (1+x)^{\frac{1}{x}} = \lim_{x\to 0}\left[1 + 1 + \frac{1-x}{2!} + \frac{(1-x)(1-2x)}{3!} + \text{..........}\infty\right]$$

$$= \left[1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \text{..........}\infty\right]$$

$$= e \quad \text{(By definition)}$$

Thus $$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

(v) Prove that

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{1}{x}\log(1+x) = \lim_{x\to 0} \log(1+x)^{1/x}$$

$$= \log e \left(\text{Using } \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \right)$$

$$= 1$$

(vi) Prove that $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof : We know that $e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\right)$

$$\therefore \quad e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots - 1\right) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\right)$$

$$\therefore \quad \frac{e^x - 1}{x} = \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\right)}{x} \qquad \text{[Dividing throughout by x]}$$

$$= \frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots\right)}{x} = \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots\right)$$

$$\therefore \quad \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots\right)$$

$$= 1 + 0 + 0 + \ldots\ldots = 1$$

Thus, $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Example 25.9 Find the value of $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

Solution : We know that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \qquad \ldots\ldots\text{(i)}$$

$\therefore$ Putting $x = -x$ in (i), we get

$$\lim_{x \to 0} \frac{e^{-x} - 1}{-x} = 1 \qquad \ldots\ldots\text{(ii)}$$

Given limit can be written as

$$\lim_{x \to 0} \frac{e^x - 1 + 1 - e^{-x}}{x} \qquad \text{[Adding (i) and (ii)]}$$

$$= \lim_{x\to 0}\left[\frac{e^x-1}{x}+\frac{1-e^{-x}}{x}\right] = \lim_{x\to 0}\left[\frac{e^x-1}{x}+\frac{e^{-x}-1}{-x}\right]$$

$$= \lim_{x\to 0}\frac{e^x-1}{x}+\lim_{x\to 0}\frac{e^{-x}-1}{-x} = 1+1 = 2 \qquad \text{[Using (i) and (ii)]}$$

Thus $\displaystyle \lim_{x\to 0}\frac{e^x-e^{-x}}{x} = 2$

Example 25.10 Evaluate : $\displaystyle \lim_{x\to 1}\frac{e^x-e}{x-1}$.

Solution : Put $x = 1 + h$, where $h \to 0$

$$\lim_{x\to 1}\frac{e^x-e}{x-1} = \lim_{h\to 0}\frac{e^{1+h}-e}{h} = \lim_{h\to 0}\frac{e^1\cdot e^h-e}{h} = \lim_{h\to 0}\frac{e(e^h-1)}{h}$$

$$= e\lim_{h\to 0}\frac{e^h-1}{h} = e\times 1 = e.$$

Thus, $\displaystyle \lim_{x\to 1}\frac{e^x-e}{x-1} = e$

Example 25.11 Evaluate : $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{x}$.

Solution : $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot 3$ [Multiplying and dividing by 3]

$$= 3\lim_{3x\to 0}\frac{\sin 3x}{3x} \quad [\because \text{ when } x\to 0, 3x\to 0]$$

$$= 3.1 \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 3$$

Thus, $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{x} = 3$

Example 25.12 Evaluate $\displaystyle \lim_{x\to 0}\frac{1-\cos x}{2x^2}$.

Solution : $\displaystyle \lim_{x\to 0}\frac{1-\cos x}{2x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2x^2}$ $\left[\begin{array}{l}\because \cos 2x = 1-2\sin^2 x,\\ \therefore 1-\cos 2x = 2\sin^2 x\\ \text{or } 1-\cos x = 2\sin^2\frac{x}{2}\end{array}\right]$

$$= \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{2 \times \frac{x}{2}} \right)^2 \quad \text{[Multiplying and dividing the denominator by 2]}$$

$$= \frac{1}{4} \lim_{\frac{x}{2} \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = \frac{1}{4} \times 1 = \frac{1}{4}$$

$$\therefore \qquad \lim_{x \to 0} \frac{1 - \cos x}{2x^2} = \frac{1}{4}$$

Example 25.13 Find the value of $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$.

Solution : Put $x = \frac{\pi}{2} + h$ $\qquad \because$ when $x \to \frac{\pi}{2}, h \to 0$

$$\therefore \qquad 2x = \pi + 2h$$

$$\therefore \qquad \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{h \to 0} \frac{1 + \cos 2\left(\frac{\pi}{2} + h \right)}{[\pi - (\pi + 2h)]^2}$$

$$= \lim_{h \to 0} \frac{1 + \cos(\pi + 2h)}{4h^2} = \lim_{h \to 0} \frac{1 - \cos 2h}{4h^2}$$

$$= \lim_{h \to 0} \frac{2 \sin^2 h}{4h^2} = \frac{1}{2} \lim_{h \to 0} \left(\frac{\sin h}{h} \right)^2 = \frac{1}{2} \times 1 = \frac{1}{2}$$

$$\therefore \qquad \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \frac{1}{2}$$

$$= \frac{a}{b}$$

$$\therefore \qquad \lim_{x \to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$$

MODULE - VIII
Calculus

Notes

CHECK YOUR PROGRESS 25.2

1. Evaluate each of the following :

(a) $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{x}$ (b) $\lim\limits_{x \to 0} \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

2. Find the value of each of the following :

(a) $\lim\limits_{x \to 1} \dfrac{e^{-x} - e^{-1}}{x - 1}$ (b) $\lim\limits_{x \to 1} \dfrac{e - e^{x}}{x - 1}$

3. Evaluate the following :

(a) $\lim\limits_{x \to 0} \dfrac{\sin 4x}{2x}$ (b) $\lim\limits_{x \to 0} \dfrac{\sin x^2}{5x^2}$ (c) $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x}$

(d) $\lim\limits_{x \to 0} \dfrac{\sin a x}{\sin b x}$

4. Evaluate each of the following :

(a) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ (b) $\lim\limits_{x \to 0} \dfrac{1 - \cos 8x}{x}$ (c) $\lim\limits_{x \to 0} \dfrac{\sin 2x(1 - \cos 2x)}{x^3}$

(d) $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{3\tan^2 x}$

5. Find the values of the following :

(a) $\lim\limits_{x \to 0} \dfrac{1 - \cos a x}{1 - \cos b x}$ (b) $\lim\limits_{x \to 0} \dfrac{x^3 \cot x}{1 - \cos x}$ (c) $\lim\limits_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x}$

6. Evaluate each of the following :

(a) $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$ (b) $\lim\limits_{x \to 1} \dfrac{\cos \frac{\pi}{2} x}{1 - x}$ (c) $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

7. Evaluate the following :

(a) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\tan 3x}$ (b) $\lim\limits_{\theta \to 0} \dfrac{\tan 7\theta}{\sin 4\theta}$ (c) $\lim\limits_{x \to 0} \dfrac{\sin 2x + \tan 3x}{4x - \tan 5x}$

25.6 CONTINUITY OF A FUNCTION AT A POINT

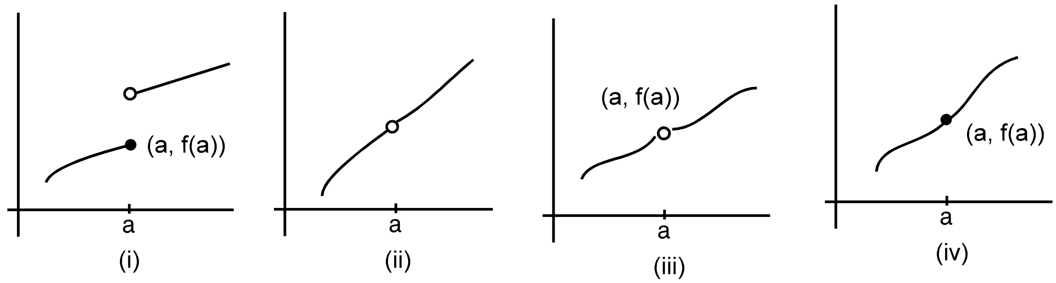


Fig. 25.5

Let us observe the above graphs of a function.

We can draw the graph (iv) without lifting the pencil but in case of graphs (i), (ii) and (iii), the pencil has to be lifted to draw the whole graph.

In case of (iv), we say that the function is continuous at x = a. In other three cases, the function is not continuous at x = a. i.e., they are discontinuous at x = a.

In case (i), the limit of the function does not exist at x = a.

In case (ii), the limit exists but the function is not defined at x = a.

In case (iii), the limit exists, but is not equal to value of the function at x = a.

In case (iv), the limit exists and is equal to value of the function at x = a.

Example 25.14 Examine the continuity of the function f (x) = $x-a$ at x = a.

Solution : $\lim\limits_{x\to a} f(x) = \lim\limits_{h\to 0} f(a+h)$

$$= \lim_{h\to 0}[(a+h)-a]$$

$$= 0 \qquad .....(i)$$

Also $\qquad f(a) = a - a = 0 \qquad .....(ii)$

From (i) and (ii),

$$\lim_{x\to a} f(x) = f(a)$$

Thus f (x) is continuous at x = a.

Example 25. 15 Show that f (x) = c is continuous.

Solution : The domain of constant function c is R.Let 'a' be any arbitrary real number.

$\therefore \qquad \lim\limits_{x\to a} f(x) = c \text{ and } f(a) = c$

$\therefore \qquad \lim\limits_{x\to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a. But 'a' is arbitrary. Hence f(x) = c is a constant function.

Example 25.16 Show that $f(x) = cx + d$ is a continuous function.

Solution : The domain of linear function $f(x) = cx + d$ is R; and let 'a' be any arbitrary real number.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to a} [c\,(a+h) + d]$$

$$= ca + d \qquad .....(i)$$

Also $\qquad f(a) = ca + d \qquad .....(ii)$

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore \quad f(x)$ is continuous at $x = a$

and $\quad$ since a is any arbitrary , $\quad f(x)$ is a continuous function.

Example 25.17 Prove that $f(x) = \sin x$ is a continuous function.

Solution : Let $f(x) = \sin x$

The domain of sin x is R. let 'a' be any arbitrary real number.

$$\therefore \quad \lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} \sin(a+h)$$

$$= \lim_{h \to 0} [\sin a . \cos h + \cos a . \sin h]$$

$$= \sin a \lim_{h \to 0} \cos h + \cos a \lim_{h \to 0} \sin h \qquad \left[\because \lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) \text{ where k is a constant}\right]$$

$$= \sin a \times 1 + \cos a \times 0 \qquad \left[\because \lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos x = 1\right]$$

$$= \sin a \qquad .....(i)$$

Also $f(a) = \sin a \qquad .....(ii)$

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore \sin x$ is continuous at $x = a$

$\because \sin x$ is continuous at $x = a$ and 'a' is an aribitary point.

Therefore, $f(x) = \sin x$ is continuous.

Definition :

1. A function $f(x)$ is said to be continuous in an open inteval]a,b[if it is continuous at every point of]a,b[*.

2. A function $f(x)$ is said to be continuous in the closed interval [a,b] if it is continuous at every point of the open interval]a,b[and is continuous at the point a from the right and continuous at b from the left.

i.e. $\lim_{x \to a^+} f(x) = f(a)$

and $\lim_{x \to b^-} f(x) = f(b)$

* *In the open interval]a,b[we do not consider the end points a and b.*

CHECK YOUR PROGRESS 25.3

1. Examine the continuity of the functions given below :

   (a) $f(x) = x - 5$ at $x = 2$ (b) $f(x) = 2x + 7$ at $x = 0$

   (c) $f(x) = \frac{5}{3}x + 7$ at $x = 3$ (d) $f(x) = px + q$ at $x = -q$

2. Show that f (x)=2a+3b is continuous, where a and b are constants.

3. Show that $5x + 7$ is a continuous function

4. (a) Show that cos x is a continuous function.

   (b) Show that cot x is continuous at all points of its domain.

5. Find the value of the constants in the functions given below :

   (a) $f(x) = px - 5$ and $f(2) = 1$ such that $f(x)$ is continuous at $x = 2$.

   (b) $f(x) = a + 5x$ and $f(0) = 4$ such that $f(x)$ is continuous at $x = 0$.

   (c) $f(x) = 2x + 3b$ and $f(-2) = \frac{2}{3}$ such that f(x) is continuous at $x = -2$.

25.7 DISCONTINUITY OF A FUNCTION AT A POINT

So far, we have considered only those functions which are continuous. Now we shall discuss some examples of functions which may or may not be continuous.

Example 25.18 Show that the function $f(x) = e^x$ is a continuous function.

Solution : Domain of e^x is R. Let $a \in R$. where 'a ' is arbitrary.

$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$, where h is a very small number.

$$= \lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a \cdot e^h = e^a \lim_{h \to 0} e^h = e^a \times 1 \qquad .....(i)$$

$$= e^a \qquad .....(ii)$$

Also $f(a) = e^a$

$\therefore$ From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a

Since a is arbitary, e^x is a continuous function.

Example 25.19 By means of graph discuss the continuity of the function $f(x)=\dfrac{x^2-1}{x-1}$.

Solution : The grah of the function is shown in the adjoining figure. The function is discontinuous as there is a gap in the graph at x = 1.

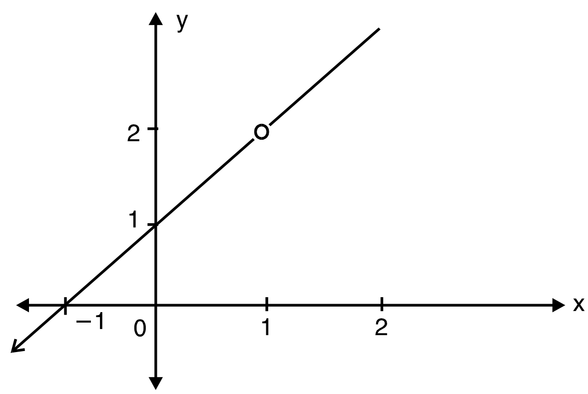


Fig. 25.6

CHECK YOUR PROGRESS 25.4

1. (a) Show that $f(x)=e^{5x}$ is a continuous function.

   (b) Show that $f(x)=e^{\frac{-2}{3}x}$ is a continuous function.

   (c) Show that $f(x)=e^{3x+2}$ is a continuous function.

   (d) Show that $f(x)=e^{-2x+5}$ is a continuous function.

2. By means of graph, examine the continuity of each of the following functions :

   (a) f (x) = x+1. (b) $f(x)=\dfrac{x+2}{x-2}$

   (c) $f(x)=\dfrac{x^2-9}{x+3}$ (d) $f(x)=\dfrac{x^2-16}{x-4}$

25.8 PROPERTIES OF CONTINUOUS FUNCTIONS

(i) Consider the function f (x) = 4. Graph of the function f (x) = 4 is shown in the Fig. 20.7. From the graph, we see that the function is continuous. In general, all constant functions are continuous.

(ii) If a function is continuous then the constant multiple of that function is also continuous.

Consider the function $f(x)=\frac{7}{2}x$. We know that x is a constant function. Let 'a' be an arbitrary real number.

$$\lim_{x\to a} f(x)=\lim_{h\to 0} f(a+h)$$

$$=\lim_{h\to 0}\frac{7}{2}(a+h)$$

$$=\frac{7}{2}a \qquad .....(i)$$

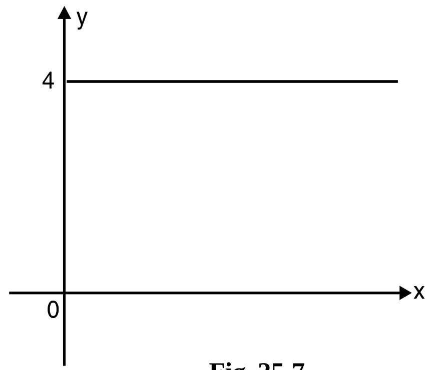


Fig. 25.7

Also $\qquad f(a)=\frac{7}{2}a \qquad .....(ii)$

$\therefore$ From (i) and (ii),

$$\lim_{x\to a} f(x)=f(a)$$

$\therefore\ f(x)=\frac{7}{2}x$ is continuous at x = a.

As $\frac{7}{2}$ is constant, and x is continuous function at x = a, $\frac{7}{2}$x is also a continuous function at x = a.

(iii) Consider the function $f(x)=x^2+2x$. We know that the function x^2 and 2x are continuous.

Now $\qquad \lim_{x\to a} f(x)=\lim_{h\to 0} f(a+h)$

$$=\lim_{h\to 0}\left[(a+h)^2+2(a+h)\right]$$

$$=\lim_{h\to 0}\left[a^2+2ah+h^2+2a+2ah\right]$$

$$=a^2+2a \qquad .....(i)$$

Also $\qquad f(a)=a^2+2a \qquad .....(ii)$

$\therefore$ From (i) and (ii), $\lim_{x\to a} f(x)=f(a)$

$\therefore$ f(x) is continuous at x = a.

Thus we can say that if x^2 and 2x are two continuous functions at x = a then $\left(x^2+2x\right)$ is also continuous at x = a.

(iv) Consider the function $f(x)=\left(x^2+1\right)(x+2)$. We know that $\left(x^2+1\right)$ and $(x+2)$ are two continuous functions.

Also $\qquad f(x)=\left(x^2+1\right)(x+2)$

$$=x^3+2x^2+x+2$$

As $x^3, 2x^2, x$ and 2 are continuous functions, therefore.

$x^3 + 2x^2 + x + 2$ is also a continuous function.

$\therefore$ We can say that if $\left(x^2+1\right)$ and (x+2) are two continuous functions then $\left(x^2+1\right)(x+2)$ is also a continuous function.

(v) Consider the function $f(x) = \dfrac{x^2-4}{x+2}$ at $x = 2$. We know that $\left(x^2-4\right)$ is continuous at $x = 2$. Also $(x + 2)$ is continuous at $x = 2$.

Again
$$\lim_{x\to 2} \frac{x^2-4}{x+2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x+2}$$
$$= \lim_{x\to 2}(x-2)$$
$$= 2 - 2 = 0$$

Also
$$f(2) = \frac{(2)^2-4}{2+2}$$
$$= \frac{0}{4} = 0$$

$\therefore$ $\lim_{x\to 2} f(x) = f(2)$. Thus $f(x)$ is continuous at $x = 2$.

$\therefore$ If $x^2 - 4$ and $x + 2$ are two continuous functions at $x = 2$, then $\dfrac{x^2-4}{x+2}$ is also continuous.

(vi) Consider the function $f(x) = |x-2|$. The function can be written as

$$f(x) = \begin{cases} -(x-2), & x < 2 \\ (x-2), & x \geq 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h), \; h > 0$$
$$= \lim_{h\to 0}\left[(2-h)-2\right]$$
$$= 2 - 2 = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h), \; h > 0 \qquad \text{......(i)}$$
$$= \lim_{x\to 2}\left[(2+h)-2\right]$$
$$= 2 - 2 = 0 \qquad \text{.....(ii)}$$

Also
$$f(2) = (2-2) = 0 \qquad \text{.....(iii)}$$

$\therefore$ From (i), (ii) and (iii), $\lim_{x\to 2} f(x) = f(2)$

Thus, $|x-2|$ is continuous at $x = 2$.

After considering the above results, we state below some properties of continuous functions.

If f (x) and g (x) are two functions which are continuous at a point x = a, then

(i) C f (x) is continuous at x = a, where C is a constant.

(ii) $f(x) \pm g(x)$ is continuous at x = a.

(iii) $f(x) \cdot g(x)$ is continuous at x = a.

(iv) f (x)/g (x) is continuous at x = a, provided g (a) $\neq$ 0.

(v) |f(x)| is continuous at x = a.

Note : Every constant function is continuous.

25.9 IMPORTANT RESULTS ON CONTINUITY

By using the properties mentioned above, we shall now discuss some important results on continuity.

(i) Consider the function $f(x) = px + q, x \in R$ (i)

The domain of this functions is the set of real numbers. Let a be any arbitary real number. Taking limit of both sides of (i), we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} (px + q) = pa + q \quad [= \text{value of } p\ x + q \text{ at } x = a.]$$

$\therefore$ px +q is continuous at x = a.

Similarly, if we consider $f(x) = 5x^2 + 2x + 3$, we can show that it is a continuous function.

In general $f(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n$

where $a_0, a_1, a_2 .....a_n$ are constants and n is a non-negative integer,

we can show that $a_0, a_1x, a_2x^2, ...a_nx^n$ are all continuos at a point x = c (where c is any real number) and by property (ii), their sum is also continuous at x = c.

$\therefore$ f (x) is continuous at any point c.

Hence every polynomial function is continuous at every point.

(ii) Consider a function $f(x) = \dfrac{(x+1)(x+3)}{(x-5)}$, $f(x)$ is not defined when $x - 5 = 0$ i.e, at x = 5.

Since (x + 1) and (x + 3) are both continuous, we can say that (x + 1) (x + 3) is also continuous. [Using property iii]

$\therefore$ Denominator of the function f (x), i.e., $(x - 5)$ is also continuous.

$\therefore$ Using the property (iv), we can say that the function $\frac{(x+1)(x+3)}{(x-5)}$ is continuous at all points except at $x = 5$.

In general if $f(x) = \frac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial functions and $q(x) \neq 0$, then f (x) is continuous if p (x) and q (x) both are continuous.

Example 25.20 Examine the continuity of the following function at $x = 2$.

$$f(x) = \begin{cases} 3x-2 & \text{for } x < 2 \\ x+2 & \text{for } x \geq 2 \end{cases}$$

Solution : Since f (x) is defined as the polynomial function $3x - 2$ on the left hand side of the point $x = 2$ and by another polynomial function $x + 2$ on the right hand side of $x = 2$, we shall find the left hand limit and right hand limit of the function at $x = 2$ separately.

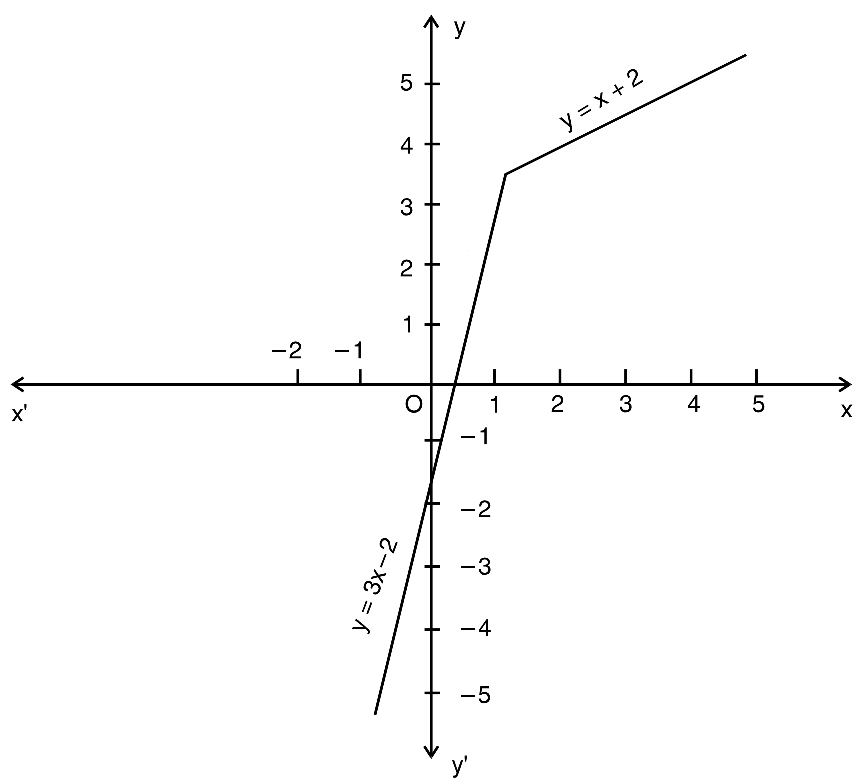


Fig. 25.8

Left hand limit $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2} (3x-2) = 3 \times 2 - 2 = 4$

Right hand limit at $x = 2$;

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(x+2) = 4$$

Since the left hand limit and the right hand limit at x = 2 are equal, the limit of the function f (x) exists at x =2 and is equal to 4 i.e., $\lim_{x\to 2} f(x) = 4$.

Also f (x) is defined by (x +2) at x = 2

$\therefore$ f (2) = 2 + 2 = 4.

Thus, $\lim_{x\to 2} f(x) = f(2)$

Hence f (x) is continuous at x = 2.

Example 25.21

(i) Draw the graph of f (x) = |x|.

(ii) Discusss the continuity of f (x) at x = 0.

Solution : We know that for $x \ge 0, |x| = x$ and for $x < 0, |x| = -x$. Hence f (x) can be written as.

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \ge 0 \end{cases}$$

(i) The graph of the function is given in Fig 20.9

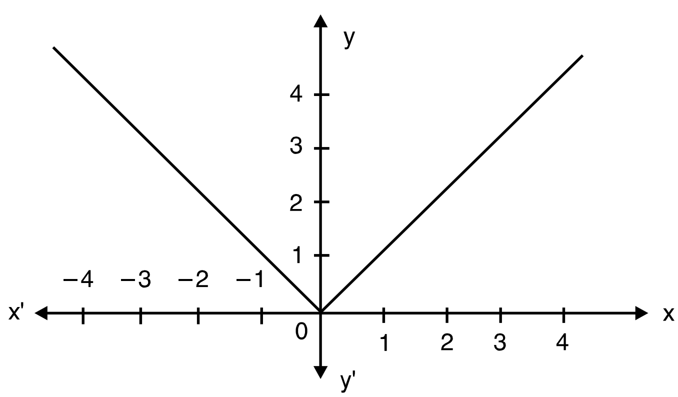


Fig. 25.9

(ii) Left hand limit $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0}(-x) = 0$

Right hand limit $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0} x = 0$

Thus, $\lim_{x\to 0} f(x) = 0$

Also, f (0) = 0

$\therefore$ $\lim_{x\to 0} f(x) = f(0)$

Hence the function $f(x)$ is continuous at $x = 0$.

Example 25.22 Examine the continuity of $f(x) = |x - b|$ at $x = b$.

Solution : We have $f(x) = |x - b|$. This function can be written as

$$f(x) = \begin{cases} -(x-b), x < b \\ (x-b), x \geq b \end{cases}$$

Left hand limit $= \lim_{x \to b^-} f(x) = \lim_{h \to 0} f(b-h)$

$$= \lim_{h \to 0} [-(b-h-b)]$$

$$= \lim_{h \to 0} h = 0 \qquad \text{.....(i)}$$

Right hand limit $= \lim_{x \to b^+} f(x) = \lim_{h \to 0} f(b+h)$

$$= \lim_{h \to 0} [(b+h)-b]$$

$$= \lim_{h \to 0} h = 0 \qquad \text{.....(ii)}$$

Also, $f(b) = b - b = 0$ .....(iii)

From (i), (ii) and (iii), $\lim_{x \to b} f(x) = f(b)$

Thus, f(x) is continuous at $x = b$.

Example 25.23 If $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

find whether f(x) is continuous at $x = 0$ or not.

Solution : Here $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

Left hand limit $= \lim_{x \to 0^-} \dfrac{\sin 2x}{x} = \lim_{h \to 0} \dfrac{\sin 2(0-h)}{0-h} = \lim_{h \to 0} \dfrac{-\sin 2h}{-h}$

$$= \lim_{h \to 0} \left(\frac{\sin 2h}{2h} \times \frac{2}{1} \right) = 1 \times 2 = 2 \qquad \text{.....(i)}$$

Right hand limit $= \lim_{x \to 0^+} \dfrac{\sin 2x}{x} = \lim_{h \to 0} \dfrac{\sin 2(0+h)}{0+h} = \lim_{h \to 0} \dfrac{\sin 2h}{2h} \times \dfrac{2}{1}$

$$= 1 \times 2 = 2 \qquad \text{(ii)}$$

Also $f(0) = 2$ (Given) (iii)

From (i) to (iii),

$$\lim_{x \to 0} f(x) = 2 = f(0)$$

Hence f(x) is continuous at x = 0.

Signum Function : The function f (x)=sgn(x) (read as signum x) is defined as

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Find the left hand limit and right hand limit of the function from its graph given below:

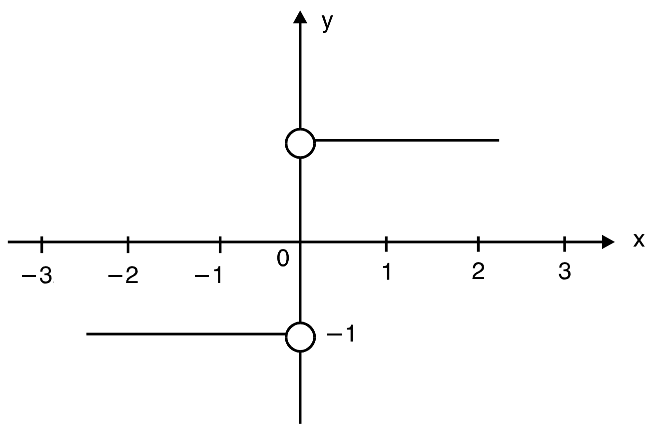


Fig. 25.11

From the graph, we see that as $x \to 0^+$, $f(x) \to 1$ and as $(x) \to 0^-$, $f(x) \to -1$

Hence, $\lim_{x \to 0^+} f(x) = 1$, $\lim_{x \to 0^-} f(x) = -1$

As these limits are not equal, $\lim_{x \to 0} f(x)$ does not exist. Hence f (x) is discontinuous at x =0.

Greatest Integer Function : Let us consider the function f (x)=[x] where [x] denotes the greatest integer less than or equal to x. Find whether f (x) is continuous at

(i) $x = \frac{1}{2}$ (ii) x =1

To solve this, let us take some arbitrary values of x say 1.3, 0.2,–0.2..... By the definition of greatest integer function,

$$[1.3] = 1, [1.99] = 1, [2] = 2, [0.2] = 0, [-0.2] = -1, [-3.1] = -4, \text{ etc.}$$

In general :

for $-3 \le x < -2$, $[x] = -3$

for $-2 \le x < -1$, $[x] = -2$

for $-1 \le x < 0$, $[x] = -1$

for $0 \le x < 1$, $[x] = 0$

for $1 \le x < 2$, $[x] = 1$ and so on.

The graph of the function f (x) = [x] is given in Fig. 25.12

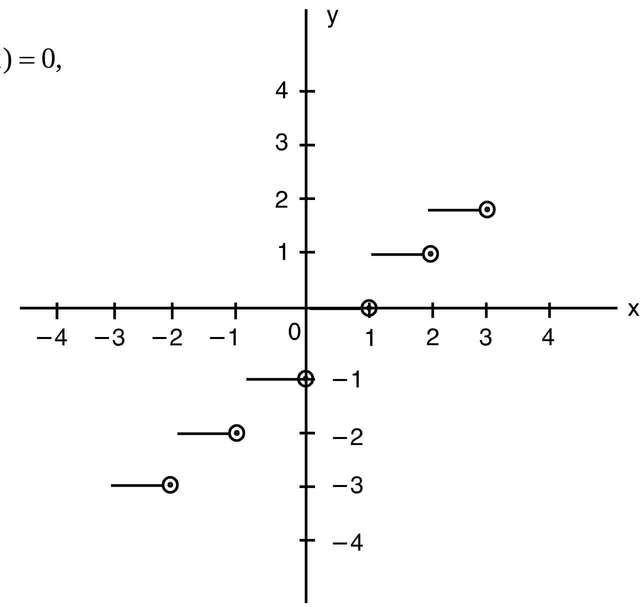


Fig. 25.12

(i) From graph

$$\lim_{x \to \frac{1}{2}^-} f(x) = 0, \quad \lim_{x \to \frac{1}{2}^+} f(x) = 0,$$

$\therefore$ $$\lim_{x \to \frac{1}{2}} f(x) = 0$$

Also $$f\left(\frac{1}{2}\right) = [0.5] = 0$$

Thus $$\lim_{x \to \frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$$

Hence f (x) is continuous at

$$x = \frac{1}{2}$$

(ii) $$\lim_{x \to 1^-} f(x) = 0, \quad \lim_{x \to 1^+} f(x) = 1$$

Thus $\lim_{x \to 1} f(x)$ does not exist.

Hence, f (x) is discontinuous at x = 1.

Note : The function f (x) = [x] is also known as Step Function.

Example 25.24 At what points is the function $\frac{x-1}{(x+4)(x-5)}$ continuous?

Solution : Here $f(x) = \frac{x-1}{(x+4)(x-5)}$

The function in the numerator i.e., x – 1 is continuous. The function in the demoninator is (x+4)(x – 5) which is also continuous.

But f (x) is not defined at the points – 4 and 5.

$\therefore$ The function f (x) is continuous at all points except – 4 and 5 at which it is not defined.

In other words, f (x) is continuous at all points of its domain.

CHECK YOUR PROGRESS 25.5

1. (a) If $f(x) = 2x + 1$, when $x \ne 1$ and f(x)=3 when x = 1, show that the function f (x) continuous at x =1.

(b) If $f(x)=\begin{cases}4x+3, & x\neq 2\\ 3x+5, & x=2\end{cases}$, find whether the function f is continuous at x = 2.

(c) Determine whether f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}4x+3, & x\leq 2\\ 8-x, & x>2\end{cases}$$

(d) Examine the continuity of f (x) at x = 1, where

$$f(x)=\begin{cases}x^2, & x\leq 1\\ x+5, & x>1\end{cases}$$

(e) Determine the values of k so that the function

$$f(x)=\begin{cases}kx^2, & x\leq 2\\ 3, & x>2\end{cases} \text{ is con tinuous at } x=2.$$

2. Examine the continuity of the following functions :

(a) $f(x)=|x-2|$ at $x=2$ (b) $f(x)=|x+5|$ at $x=-5$

(c) $f(x)=|a-x|$ at $x=a$

(d) $f(x)=\begin{cases}\dfrac{|x-2|}{x-2}, & x\neq 2\\ 1, & x=2\end{cases}$ at x = 2

(e) $f(x)=\begin{cases}\dfrac{|x-a|}{x-a}, & x\neq a\\ 1, & x=a\end{cases}$ at x = a

3. (a) If $f(x)=\begin{cases}\sin 4x, & x\neq 0\\ 2, & x=0\end{cases}$, at x = 0

(b) If $f(x)=\begin{cases}\dfrac{\sin 7x}{x}, & x\neq 0\\ 7, & x=0\end{cases}$, at x = 0

(c) For what value of a is the function

$$f(x)=\begin{cases}\dfrac{\sin 5x}{3x}, & x\neq 0\\ a, & x=0\end{cases} \text{ continuous at } x=0\,?$$

4. (a) Show that the function f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2}, & \text{for } x\neq 2\\ 3, & \text{for } x=2\end{cases}$$

(b) Test the continuity of the function f (x) at x = 1, where

$$f(x)=\begin{cases}\dfrac{x^2-4x+3}{x-1} & \text{for } x\neq 1\\ -2 & \text{for } x=1\end{cases}$$

(c) For what value of k is the following function continuous at x = 1?

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & \text{when } x\neq 1\\ k & \text{when } x=1\end{cases}$$

(d) Discuss the continuity of the function f (x) at x = 2, when

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{for } x\neq 2\\ 7, & x=2\end{cases}$$

5. (a) If $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$, find whether f is continuous at x = 0.

(b) Test the continuity of the function f (x) at the origin.

where $$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 1, & x=0\end{cases}$$

6. Find whether the function f (x)=[x] is continuous at

(a) $x=\frac{4}{3}$ (b) $x=3$ (c) $x=-1$ (d) $x=\frac{2}{3}$

7. At what points is the function f (x) continuous in each of the following cases ?

(a) $f(x)=\dfrac{x+2}{(x-1)(x-4)}$ (b) $f(x)=\dfrac{x-5}{(x+2)(x-3)}$ (c) $f(x)=\dfrac{x-3}{x^2+5x-6}$

(d) $f(x)=\dfrac{x^2+2x+5}{x^2-8x+16}$

LET US SUM UP

- If a function f(x) approaches l when x approches a, we say that l is the limit of f (x). Symbolically, it is written as

$$\lim_{x\to a} f(x)=\ell$$

- If $\lim_{x\to a} f(x)=\ell$ and $\lim_{x\to a} g(x)=m$, then

(i) $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) = k\ell$

(ii) $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = \ell \pm m$

(iii) $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = \ell m$

(iv) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{\ell}{m}$, provided $\lim_{x\to a} g(x) \neq 0$

- **LIMIT OF IMPORTANT FUNCTIONS**

(i) $\lim_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x\to 0} \sin x = 0$

(iii) $\lim_{x\to 0} \cos x = 1$

(iv) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

(v) $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(vi) $\lim_{x\to 0} \dfrac{\log(1+x)}{x} = 1$

(vii) $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=HB8CzZEd4xw

http://www.zweigmedia.com/RealWorld/Calcsumm3a.html

http://www.intuitive-calculus.com/limits-and-continuity.html

TERMINAL EXERCISE

Evaluate the following limits :

1. $\lim_{x\to 1} 5$

2. $\lim_{x\to 0} \sqrt{2}$

3. $\lim_{x\to 1} \dfrac{4x^5 + 9x + 7}{3x^6 + x^3 + 1}$

4. $\lim_{x\to -2} \dfrac{x^2 + 2x}{x^3 + x^2 - 2x}$

5. $\lim_{x\to 0} \dfrac{(x+k)^4 - x^4}{k(k+2x)}$

6. $\lim_{x\to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$

7. $\lim\limits_{x \to -1} \left[\dfrac{1}{x+1} + \dfrac{2}{x^2-1} \right]$

8. $\lim\limits_{x \to 1} \dfrac{(2x-3)\sqrt{x}-1}{(2x+3)(x-1)}$

9. $\lim\limits_{x \to 2} \dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

10. $\lim\limits_{x \to 1} \left[\dfrac{1}{x-1} - \dfrac{2}{x^2-1} \right]$

11. $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$

12. $\lim\limits_{x \to a} \dfrac{x^2-(a+1)x+a^2}{x^2-a^2}$

Find the left hand and right hand limits of the following functions :

13. $f(x) = \begin{cases} -2x+3 \text{ if } x \le 1 \\ 3x-5 \quad \text{if } x > 1 \end{cases} \text{as } x \to 1$

14. $f(x) = \dfrac{x^2-1}{|x+1|} \text{as } x \to 1$

Evaluate the following limits :

15. $\lim\limits_{x \to 1^-} \dfrac{|x+1|}{x+1}$

16. $\lim\limits_{x \to 2^+} \dfrac{|x-2|}{x-2}$

17. $\lim\limits_{x \to 2^-} \dfrac{x-2}{|x-2|}$

18. If $f(x) = \dfrac{(x+2)^2-4}{x}$, prove that $\lim\limits_{x \to 0} f(x) = 4$ though $f(0)$ is not defined.

19. Find k so that $\lim\limits_{x \to 2} f(x)$ may exist where $f(x) = \begin{cases} 5x+2, x \le 2 \\ 2x+k, x > 2 \end{cases}$

20. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 7x}{2x}$

21. Evauate $\lim\limits_{x \to 0} \left[\dfrac{e^x + e^{-x} - 2}{x^2} \right]$

22. Evaluate $\lim\limits_{x \to 0} \dfrac{1-\cos 3x}{x^2}$

23. Find the value of $\lim\limits_{x \to 0} \dfrac{\sin 2x + 3x}{2x + \sin 3x}$

24. Evaluate $\lim\limits_{x \to 1} (1-x) \tan \dfrac{\pi x}{2}$

25. Evaluate $\lim\limits_{\theta \to 0} \frac{\sin 5\theta}{\tan 8\theta}$

Examine the continuity of the following :

26. $f(x)\begin{cases} 1+3x \text{ if } x>-1 \\ 2 \text{ if } x \le -1 \end{cases}$

at $x=-1$

27. $f(x)=\begin{cases} \frac{1}{x}-x, 0<x<\frac{1}{2} \\ \frac{1}{2}, x=\frac{1}{2} \\ \frac{3}{2}-x, \frac{1}{2}<x<1 \end{cases}$

at $x=\frac{1}{2}$

28. For what value of k, will the function

$$f(x)=\begin{cases} \frac{x^2-16}{x-4} \text{ if } x \ne 4 \\ k \text{ if } x=4 \end{cases}$$

be continuous at $x = 4$?

29. Determine the points of discontinuty, if any, of the following functions :

(a) $\frac{x^2+3}{x^2+x+1}$ (b) $\frac{4x^2+3x+5}{x^2-2x+1}$

(b) $\frac{x^2+x+1}{x^2-3x+1}$ (d) $f(x)=\begin{cases} x^4-16, & x \ne 2 \\ 16, & x=2 \end{cases}$

30. Show that the function $f(x)=\begin{cases} \frac{\sin x}{x}+\cos, & x \ne 0 \\ 2, & x=0 \end{cases}$ is continuous at $x = 0$

31. Determine the value of 'a', so that the function f (x) defined by

$$f(x)=\begin{cases} \frac{a\cos x}{\pi-2x}, & x \ne \frac{\pi}{2} \\ 5, & x=\frac{\pi}{2} \end{cases}$$

is continuous.

ANSWERS

CHECK YOUR PROGRESS 25.1

1. (a) 17 (b) 7 (c) 0 (d) 2
   (e) - 4 (f) 8

2. (a) 0 (b) $\frac{3}{2}$ (c) $-\frac{2}{11}$ (d) $\frac{q}{b}$ (e) 6
   (f) -10 (g) 3 (h) 2

3. (a) 3 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{1}{2}$

4. (a) $\frac{1}{2}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{2\sqrt{6}}$ (d) 2 (e) -1

5. (a) Does not exist (b) Does not exist

6. (a) 0 (b) $\frac{1}{4}$ (c) does not exist

7. (a) $1, -2$ (b) 1 (c) 19

8. $a = -2$

10. limit does not exist

CHECK YOUR PROGRESS 25.2

1. (a) 2 (b) $\frac{e^2 - 1}{e^2 + 1}$

2. (a) $-\frac{1}{e}$ (b) $-e$

3. (a) 2 (b) $\frac{1}{5}$ (c) 0 (d) $\frac{a}{b}$

4. (a) $\frac{1}{2}$ (b) 0 (c) 4 (d) $\frac{2}{3}$

5. (a) $\frac{a^2}{b^2}$ (b) 2 (c) $\frac{1}{2}$

6. (a) 1 (b) $\frac{\pi}{2}$ (c) 0

7. (a) $\frac{5}{3}$ (b) $\frac{7}{4}$ (c) -5

CHECK YOUR PROGRESS 25.3

1. (a) Continuous (b) Continuous
   (c) Continuous (d) Continuous

5. (a) p =3 (b) a = 4
   (c) $b = \frac{14}{9}$

CHECK YOUR PROGRESS 25.4

2. (a) Continuous
   (b) Discontinuous at x = 2
   (c) Discontinuous at $x = -3$
   (d) Discontinuous at x = 4

CHECK YOUR PROGRESS 25.5

1. (b) Continuous (c) Discontinuous
   (d) Discontinuous (e) $k = \frac{3}{4}$

2 (a) Continuous (c) Continuous,
  (d) Discontinuous (e) Discontinuous

3 (a) Discontinuous (b) Continuous (c) $\frac{5}{3}$

4 (b) Continuous (c) k = 2
  (d) Discontinuous

5. (a) Discontinuous (b) Discontinuous

6 (a) Continuous (b) Discontinuous
  (c) Discontinuous (d) Continuous

7. (a) All real number except 1 and 4
   (b) All real numbers except –2 and 3
   (c) All real number except –6 and 1
   (d) All real numbers except 4

TERMINAL EXERCISE

1. 5	2. $\sqrt{2}$	3. 4	4. $-\frac{1}{3}$
5. $2x^2$	6. 1	7. $-\frac{1}{2}$	8. $-\frac{1}{10}$

9.	-8	10.	$\frac{1}{2}$
11.	1	12.	$\frac{a-1}{2a}$
13.	$1, -2$	14.	$-2, 2$
15.	-1	16.	1
17.	-1	19.	$k = 8$
20.	$\frac{7}{2}$	21.	1
22.	$\frac{9}{2}$	23.	1
24.	$\frac{2}{\pi}$	25.	$\frac{5}{8}$

26. Discontinuous

27. Discontinuous

28. $k = 8$

29. (a) No (b) $x = 1$

(c) $x = 1, x = 2$ (d) $x = 2$

31. 10

26

DIFFERENTIATION

The differential calculus was introduced sometime during 1665 or 1666, when Isaac Newton first concieved the process we now know as differentiation (a mathematical process and it yields a result called derivative). Among the discoveries of Newton and Leibnitz are rules for finding derivatives of sums, products and quotients of composite functions together with many other results. In this lesson we define derivative of a function, give its geometrical and physical interpretations, discuss various laws of derivatives and introduce notion of second order derivative of a function.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the derivative of a function y = f(x) at x = a;
- prove that the derivative of a constant function f(x) = c, is zero;
- find the derivative of $f(x) = x^n, n \in Q$ from first principle and apply to find the derivatives of various functions;
- find the derivatives of the functions of the form cf(x), $[f(x) \pm g(x)]$ and polynomial functions;
- state and apply the results concerning derivatives of the product and quotient of two functions;
- state and apply the chain rule for the derivative of a function;
- find the derivative of algebraic functions (including rational functions); and
- find second order derivative of a function.

EXPECTED BACKGROUND KNOWLEDGE

- Binomial Theorem
- Functions and their graphs
- Notion of limit of a function

26.1 DERIVATIVE OF A FUNCTION

Consider a function and a point say (5,25) on its graph. If x changes from 5 to 5.1, 5.01, 5.001..... etc., then correspondingly, y changes from 25 to 26.01, 25.1001, 25.010001,....A small change in x causes some small change in the value of y. We denote this change in the value of x by a symbol δx and the corresponding change caused in y by δy and call these respectively as an increment in x and increment in y, irrespective of sign of increment. The ratio $\frac{\delta x}{\delta y}$ of increment is termed as incrementary ratio. Here, observing the following table for $y = x^2$ at (5,25), we have for $\delta x = 0.1, 0.01, 0.001, 0.0001, \ldots$ $\delta y = 1.01, .1001, .010001, .00100001, \ldots$

x	5.1	5.01	5.001	5.0001
δx	.1	.01	.001	.0001
y	26.01	25.1001	25.010001	25.00100001
δy	1.01	.1001	.010001	.00100001
$\frac{\delta y}{\delta x}$	10.1	10.01	10.001	10.0001

We make the following observations from the above table :

(i) δy varies when δx varies.

(ii) $\delta y \to 0$ when $\delta x \to 0$.

(iii) The ratio $\frac{\delta y}{\delta x}$ tends to a number which is 10.

Hence, this example illustrates that $\delta y \to 0$ when $\delta x \to 0$ but $\frac{\delta y}{\delta x}$ tends to a finite number, not necessarily zero. The limit, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is equivalently represented by $\frac{dy}{dx}$. $\frac{dy}{dx}$ is called the derivative of y with respect to x and is read as differential coefficient of y with respect to x.

That is, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = 10$ in the above example and note that while δx and δy are small numbers (increments), the ratio $\frac{\delta y}{\delta x}$ of these small numbers approaches a definite value 10.

In general, let us consider a function

$$y = f(x) \qquad \text{...(i)}$$

To find its derivative, consider δx to be a small change in the value of x, so $x + \delta x$ will be the new value of x where $f(x)$ is defined. There shall be a corresponding change in the value of y. Denoting this change by δy ; $y + \delta y$ will be the resultant value of y, thus,

$$y + \delta y = f(x + \delta x) \qquad \text{...(ii)}$$

Subtracting (i) from (ii), we have,

$$(y + \delta y) - y = f(x + \delta x) - f(x)$$

or

$$\delta y = f(x + \delta x) - f(x) \qquad \text{...(iii)}$$

To find the rate of change, we divide (iii) by δx

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} \qquad \text{...(iv)}$$

Lastly, we consider the limit of the ratio $\frac{\delta y}{\delta x}$ as $\delta x \to 0$.

If $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} \qquad \text{...(v)}$$

is a finite quantity, then f(x) is called derivable and the limit is called derivative of f(x) with respect to (w.r.t.) x and is denoted by the symbol $f'(x)$ or by $\frac{d}{dx}$ of f(x)

i.e. $\frac{d}{dx}f(x)$ or $\frac{dy}{dx}$ (read as $\frac{d}{dx}$ of y).

Thus, $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\frac{dy}{dx}=\frac{d}{dx}f(x)=f'(x)$$

Remarks

(1) The limiting process indicated by equation (v) is a mathematical operation. This mathematical process is known as differentiation and it yields a result called a derivative.

(2) A function whose derivative exists at a point is said to be derivable at that point.

(3) It may be verified that if f(x) is derivabale at a point x = a, then, it must be continuous at that point. However, the converse is not necessarily true.

(4) The symbols Δx and h are also used in place of δx i.e.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{or} \quad \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

(5) If y = f(x), then $\frac{dy}{dx}$ is also denoted by y_1 or y'.

26.2 VELOCITY AS LIMIT

Let a particle initially at rest at 0 moves along a strainght line OP., The distance s

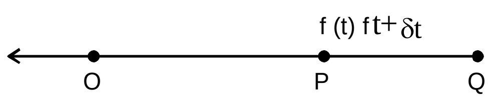


Fig. 26.1

covered by it in reaching P is a function of time t, We may write distance

$$OP = s = f(t) \qquad \text{...(i)}$$

In the same way in reaching a point Q close to P covering PQ

i.e., δs is a fraction of time δt so that

$$OQ = OP + PQ$$

$$= s + \delta s$$

$$= f(t+\delta t) \qquad \text{...(ii)}$$

The average velocity of the particle in the interval δt is given by

$$= \frac{\text{Change in distance}}{\text{Change in time}}$$

$$= \frac{(s+\delta s) - s}{(t+\delta t) - t}, \qquad \text{[From (i) and (ii)]}$$

$$= \frac{f\,(t+\delta t) - f(t)}{\delta t}$$

(average rate at which distance is travelled in the interval δt).

Now we make δt smaller to obtain average velocity in smaller interval near P. The limit of average velocity as $\delta t \to 0$ is the instantaneous velocity of the particle at time t (at the point P).

$\therefore$ Velocity at time $t = \lim\limits_{\delta t \to 0} \dfrac{f\,(t+\delta t) - f\,(t)}{\delta t}$

It is denoted by $\dfrac{ds}{dt}$.

Thus, if f (t) gives the distance of a moving particle at time t, then the derivative of 'f' at $t = t_0$ represents the instantaneous speed of the particle at the point P i.e. at time $t = t_0$.

This is also referred to as the physical interpretation of a derivative of a function at a point.

Note : The derivative $\dfrac{dy}{dx}$ represents instantaneous rate of change of y w.r.t. x.

Example 26.1 The distance 's' meters travelled in time t seconds by a car is given by the relation

$$s = 3t^2$$

Find the velocity of car at time t =4 seconds.

Solution : Here, $f\,(t) = s = 3t^2$

$$\therefore \qquad f\,(t+\delta t) = s + \delta s = 3(t+\delta t)^2$$

Velocity of car at any time $t = \lim\limits_{\delta t \to 0} \dfrac{f\,(t+\delta t) - f\,(t)}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{3(t+\delta t)^2 - 3t^2}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{3(t^2 + 2t.\delta t + \delta t^2) - 3t^2}{\delta t}$$

$$= \lim_{\delta t \to 0} (6t + 3\delta t)$$

$$= 6t$$

$\therefore$ Velocity of the car at t = 4 sec = (6×4) m/sec = 24 m/sec.

CHECK YOUR PROGRESS 26.1

1. Find the velocity of particles moving along a straight line for the given time-distance relations at the indicated values of time t :

(a) $s = 2 + 3t; t = \frac{1}{3}.$ (b) $s = 8t - 7; t = 4.$

(c) $s = t^2 + 3t; t = \frac{3}{2}.$ (d) $s = 7t^2 - 4t + 1; t = \frac{5}{2}.$

2. The distance s metres travelled in t seconds by a particle moving in a straight line is given by $s = t^4 - 18t^2$. Find its speed at t = 10 seconds.

3. A particle is moving along a horizontal line. Its distance s meters from a fixed point O at t seconds is given by $s = 10 - t^2 + t^3$. Determine its instantaneous speed at the end of 3 seconds.

26.3 GEOMETRICAL INTERPRETATION OF dy/dx

Let y = f (x) be a continuous function of x, draw its graph and denote it by APQB.

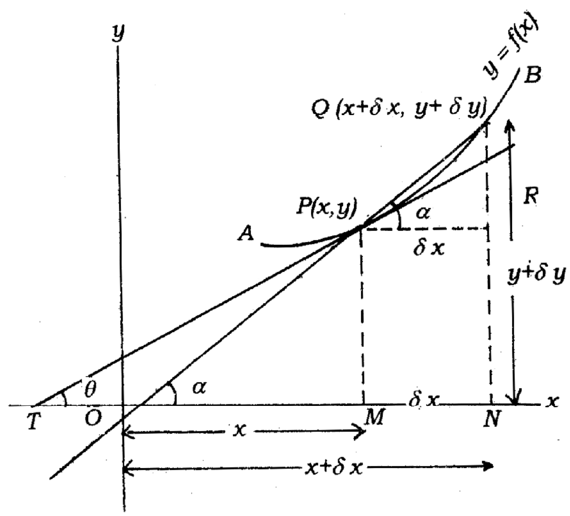


Fig. 26.2

Let P (x,y) be any point on the graph of y = f(x) or curve represented by y = f(x). Let $Q(x + \delta x, y + \delta y)$ be another point on the same curve in the neighbourhood of point P.

Draw PM and QN perpendiculars to x-axis and PR parallel to x-axis such that PR meets QN at R.Join QP and produce the secant line to any point S. Secant line QPS makes angle say α with the positive direction of x-axis. Draw PT tangent to the curve at the point P, making angle θ with the x-axis.

Now, In $\Delta QPR, \angle QPR = \alpha$

$$\tan\alpha = \frac{QR}{PR} = \frac{QN - RN}{MN} = \frac{QN - PM}{ON - OM} = \frac{(y + \delta y) - y}{(x + \delta x) - x} = \frac{\delta y}{\delta x} \qquad \text{(i)}$$

Now, let the point Q move along the curve towards P so that Q approaches nearer and nearer the point P.

Thus, when $Q \to P$, $\delta x \to 0$, $\delta y \to 0$, $\alpha \to 0, (\tan\alpha \to \tan\theta)$ and consequently, the secant QPS tends to coincide with the tangent PT.

From (i). $$\tan\alpha = \frac{\delta y}{\delta x}$$

In the limiting case, $$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \alpha \to 0}} \tan\alpha = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

or $$\tan\theta = \frac{dy}{dx} \qquad ...(ii)$$

Thus the derivative $\frac{dy}{dx}$ of the function $y = f(x)$ at any point P (x,y) on the curve represents the slope or gradient of the tangent at the point P.

This is called the geometrical interpretation of $\frac{dy}{dx}$.

It should be noted that $\frac{dy}{dx}$ has different values at different points of the curve.

Therefore, in order to find the gradient of the curve at a particular point, find $\frac{dy}{dx}$ from the equation of the curve $y = f(x)$ and substitute the coordinates of the point in $\frac{dy}{dx}$.

Corollary 1

If tangent to the curve at P is parallel to x-axis, then $\theta = 0°$ or180°, i.e., $\frac{dy}{dx} = \tan 0°$ or tan 180° i.e., $\frac{dy}{dx} = 0$.

That is tangent to the curve represented by $y = f(x)$ at P is parallel to x-axis.

Corollary 2

If tangent to the curve at P is perpendicular to x-axis, $\theta = 90°$ or $\frac{dy}{dx} = \tan 90° = \infty$.

That is, the tangent to the curve represented by $y = f(x)$ at P is parallel to y-axis.

26.4 DERIVATIVE OF CONSTANT FUNCTION

Statement : The derivative of a constant is zero.

Proof : Let y = c be a constant function. Then y = c can be written as

$$y = cx^0 \qquad [\because x^0 = 1] \qquad ..(i)$$

Let δx be a small increment in x. Corresponding to this increment, let δy be the increment in the value of y so that

$$y + \delta y = c(x + \delta x)^0 \qquad ..(ii)$$

Subtracting (i) from (ii),

$$(y + \delta y) - y = c(x + \delta x)^0 - cx^0, \qquad (\because x^0 = 1)$$

or $$\delta y = c - c \quad \text{or} \quad \delta y = 0$$

Dividing by δx, $$\frac{\delta y}{\delta x} = \frac{0}{\delta x} \quad \text{or} \quad \frac{\delta y}{\delta x} = 0$$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 0 \quad \text{or} \quad \frac{dy}{dx} = 0$$

or $$\frac{dc}{dx} = 0 \qquad [\, y = c \text{ from (i)}]$$

This proves that rate of change of constant quantity is zero. Therefore, derivative of a constant quantity is zero.

26.5 DERIVATIVE OF A FUNCTION FROM FIRST PRINCIPLE

Recalling the definition of derivative of a function at a point, we have the following working rule for finding the derivative of a function from first principle:

Step I. Write down the given function in the form of $y = f(x)$ ....(i)

Step II. Let dx be an increment in x, δy be the corresponding increment in y so that

$$y + \delta y = f(x + \delta x) \qquad ....(ii)$$

Step III. Subtracting (i) from (ii), we get

$$\delta y = f(x + \delta x) - f(x) \qquad ..(iii)$$

Step IV. Dividing the result obtained in step (iii) by δx, we get,

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Step V. Proceeding to limit as $\delta x \to 0$.

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Note : The method of finding derivative of function from first principle is also called **delta** or **ab-ininitio** method.

Next, we find derivatives of some standard and simple functions by first principle.

$$\text{Let } y = x^n \qquad .....(i)$$

For a small increment δx in x, let the corresponding increment in y be δy.

Then $$y + \delta y = (x + \delta x)^n. \qquad ...(ii)$$

Subtracing (i) from (ii) we have,

$$(y + \delta y) - y = (x + \delta x)^n - x^n$$

$$\therefore \qquad \delta y = x^n \left(1 + \frac{\delta x}{x}\right)^n - x^n$$

$$= x^n \left[\left(1 + \frac{\delta x}{x}\right)^n - 1\right]$$

Since $\frac{\delta x}{x} < 1$, as δx is a small quantity compared to x, we can expand $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem for any index.

Expanding $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem, we have

$$\delta y = x^n \left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{\delta x}{x}\right)^3 + \ldots - 1\right]$$

$$= x^n (\delta x)\left[\frac{n}{x} + \frac{n(n-1)}{2}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

Proceeding to limit when $\delta x \to 0$, $(\delta x)^2$ and higher powers of δx will also tend to zero.

$$\therefore \qquad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3} + \ldots\right]$$

or $$\operatorname{lt}_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = x^n \left[\frac{n}{x} + 0 + 0 + \ldots\right]$$

or $$\frac{dy}{dx} = x^n \cdot \frac{n}{x} = nx^{n-1}$$

or $$\mathbf{\frac{d}{dx}(x^n) = nx^{n-1},} \qquad \left[\because \; y = x^n\right]$$

This is known as Newton's Power Formula or Power Rule

Note : We can apply the above formula to find derivative of functions like $x, x^2, x^3, ...$

i.e. when $n = 1,2,3,...$

e.g. $$\frac{d}{dx}x = \frac{d}{dx}x^1 = 1x^{1-1} = 1x^0 = 1.1 = 1$$

$$\frac{d}{dx}x^2 = 2x^{2-1} = 2x$$

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2, \text{ and so on.}$$

Example 26.2 Find the derivative of each of the following :

(i) x^{10} (ii) x^{50} (iii) x^{91}

Solution :

(i) $$\frac{d}{dx}\left(x^{10}\right) = 10x^{10-1} = 10x^9$$

(ii) $$\frac{d}{dx}\left(x^{50}\right) = 50x^{50-1} = 50x^{49}$$

(iii) $$\frac{d}{dx}\left(x^{91}\right) = 91x^{91-1} = 91x^{90}$$

We shall now find the derivatives of some simple functions from definition or first principles.

Example 26.3 Find the derivative of x^2 from the first principles.

Solution : Let $$y = x^2 \qquad \text{(i)}$$

For a small increment δx in x let the corresponding increment in y be δy.

$$y + \delta y = (x + \delta x)^2 \qquad \text{(ii)}$$

Subtracting (i) from (ii), we have

$$(y + \delta y) - y = (x + \delta x)^2 - x^2$$

or $$\delta y = x^2 + 2x(\delta x) + (\delta x)^2 - x^2$$

or $$\delta y = 2x(\delta x) + (\delta x)^2$$

Divide by δx, we have

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

Proceeding to limit when $\delta x \to 0$, we have

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(2x + \delta x)$$

or $$\frac{dy}{dx} = 2x + \lim_{\delta x \to 0}(\delta x)$$

$$= 2x + 0$$
$$= 2x$$

or $$\frac{dy}{dx} = 2x \quad \text{or} \quad \frac{d}{dx}\left(x^2\right) = 2x$$

or $$\frac{dy}{dx} = \frac{-1}{x(x+0)} \quad \text{or} \quad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Example 26.4 Find the derivative of $\sqrt{x}$ by ab-initio method.

Solution : Let $y = \sqrt{x}$ ..(i)

For a small increment δx in x, let δy be the corresponding increment in y.

$\therefore$ $$y + \delta y = \sqrt{x + \delta x}$$ ...(ii)

Subtracting (i) from (ii), we have

$$(y+\delta y) - y = \sqrt{x+\partial x} - \sqrt{x}$$ ...(iii)

or $$\delta y = \sqrt{x + \delta x} - \sqrt{x}$$

Rationalising the numerator of the right hand side of (iii), we have

$$\delta y = \frac{\sqrt{x+\delta x} - \sqrt{x}}{\sqrt{x+\delta x} + \sqrt{x}}\left(\sqrt{x+\delta x} + \sqrt{x}\right)$$

$$= \frac{(x+\delta x) - x}{\sqrt{x+\delta x + \sqrt{x}}} \quad \text{or} \quad \delta y = \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}}$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$$

Proceeding to limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[\frac{1}{\sqrt{x+\delta} + \sqrt{x}}\right]$$

or $$\frac{dy}{dx} = \frac{1}{\sqrt{x} + \sqrt{x}} \quad \text{or} \quad \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

Example 26.5 If f (x) is a differentiable function and c is a constant, find the derivative of

$$\phi(x) = cf(x)$$

Solution : We have to find derivative of function

$$\phi(x) = cf(x)$$ ...(i)

For a small increment δx in x, let the values of the functions $\phi(x)$ be $\phi(x+\delta x)$ and that of f (x)be $f(x+\delta x)$

$\therefore$ $$\phi(x+\delta x) = cf(x+\delta x)$$ ...(ii)

Subtracting (i) from (ii), we have

$$\phi(x+\delta x)-\phi(x)=c\left[f(x+\delta x)-f(x)\right]$$

Dividing by δx, we have

$$\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=c\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

Proceeding to limit as $\delta x \to 0$, we have

$$\lim_{\delta x\to 0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x}=\lim_{\delta x\to 0} c\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

or $$\phi'(x)=c\lim_{\delta x\to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

or $$\phi'(x)=cf'(x)$$

Thus, $$\frac{d}{dx}\left[cf\left(x\right)\right]=c\frac{df}{dx}$$

CHECK YOUR PROGRESS 26.2

1. Find the derivative of each of the following functions by delta method :

   (a) $10x$ (b) $2x+3$ (c) $3x^2$

   (d) x^2+5x (e) $7x^3$

2. Find the derivative of each of the following functions using ab-initio method:

   (a) $\frac{1}{x}, x\neq 0$ (b) $\frac{1}{ax}, x\neq 0$ (c) $x+\frac{1}{x}, x\neq 0$

   (d) $\frac{1}{ax+b}, x\neq\frac{-b}{a}$ (e) $\frac{ax+b}{cx+d}, x\neq\frac{-d}{c}$ (f) $\frac{x+2}{3x+5}, x\neq\frac{-5}{3}$

3. Find the derivative of each of the following functions from first principles :

   (a) $\frac{1}{\sqrt{x}}, x\neq 0$ (b) $\frac{1}{\sqrt{ax+b}}, x\neq\frac{-b}{a}$ (c) $\sqrt{x}+\frac{1}{\sqrt{x}}, x\neq 0$

   (d) $\frac{1+x}{1-x}, x\neq 1$

4. Find the derivative of each of the following functions by using delta method :

   (a) $f(x)=3\sqrt{x}$. Also find $f'(2)$. (b) $f(r)=\pi r^2$. Also find $f'(2)$.

   (c) $f(r)=\frac{4}{3}\pi r^3$. Also find $f'(3)$.

26.7 ALGEBRA OF DERIVATIVES

Many functions arise as combinations of other functions. The combination could be sum, difference, product or quotient of functions. We also come across situations where a given function can be expressed as a function of a function.

In order to make derivative as an effective tool in such cases, we need to establish rules for finding derivatives of sum, difference, product, quotient and function of a function. These, in turn, will enable one to find derivatives of polynomials and algebraic (including rational) functions.

26.7 DERIVATIVES OF SUM AND DIFFERENCE OF FUNCTIONS

If f (x) and g (x) are both derivable functions and h (x) = f (x) + g (x), then what is h' (x) ?

Here $\qquad h(x) = f(x) + g(x)$

Let δx be the increment in x and δy be the correponding increment in y.

$$\therefore \qquad h(x+\delta x) = f(x+\delta x) + g(x+\delta x)$$

Hence
$$h'(x) = \lim_{\delta x \to 0} \frac{[f(x+\delta x) + g(x+\delta x)] - [f(x) + g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[f(x+\delta x) - f(x)] + [g(x+\delta x) - g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x} + \frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} + \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

or
$$h'(x) = f'(x) + g'(x)$$

Thus we see that the *derivative of sum of two functions is sum of their derivatives*.

This is called the **SUM RULE**.

e.g. $\qquad y = x^2 + x^3$

Then
$$y' = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(x^3\right)$$

$$= 2x + 3x^2$$

Thus $\qquad y' = 2x + 3x^2$

This sum rule can easily give us the difference rule as well, because

if $\qquad h(x) = f(x) - g(x)$

then $\qquad h(x) = f(x) + [-g(x)]$

$$\therefore \qquad h'(x) = f'(x) + [-g'(x)]$$

$$= f'(x) - g'(x)$$

i.e. *the derivative of difference of two functions is the difference of their derivatives*.

This is called **DIFFERENCE RULE.**
Thus we have

Sum rule : $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Difference rule : $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Example 26.6 Find the derivative of each of the following functions :

(i) $y = 10t^2 + 20t^3$

(ii) $y = x^3 + \frac{1}{x^2} - \frac{1}{x}, \quad x \neq 0$

Solution :

(i) We have, $y = 10t^2 + 20t^3$

$$\therefore \qquad \frac{dy}{dt} = 10(2t) + 20(3t^2)$$

$$= 20t + 60t^2$$

(ii) $y = x^3 + \frac{1}{x^2} - \frac{1}{x} \qquad\qquad x \neq 0$

$$= x^3 + x^{-2} - x^{-1}$$

$$\therefore \qquad \frac{dy}{dx} = 3x^2 + (-2)x^{-3} - (-1)x^{-2} \;\; = 3x^2 - \frac{2}{x^3} + \frac{1}{x^2}$$

Example 26.7 Evaluate the derivative of

$$y = x^3 + 3x^2 + 4x + 5, \;\; x = 1$$

Solution :

(ii) We have $y = x^3 + 3x^2 + 4x + 5$

$$\therefore \qquad \frac{dy}{dx} = \frac{d}{dx}\left[x^3 + 3x^2 + 4x + 5\right] = 3x^2 + 6x + 4$$

$$\therefore \qquad \left.\frac{dy}{dx}\right]_{x=1} = 3(1)^2 + 6(1) + 4 \;= 13$$

CHECK YOUR PROGRESS 26.3

1. Find y' when :

   (a) $y = 12$ (b) $y = 12x$ (c) $y = 12x + 12$

2. Find the derivatives of each of the following functions :

   (a) $f(x) = 20x^9 + 5x$ (b) $f(x) = -50x^4 - 20x^2 + 4$

   (c) $f(x) = 4x^3 - 9 - 6x^2$ (d) $f(x) = \frac{5}{9}x^9 + 3x$

   (e) $f(x) = x^3 - 3x^2 + 3x - \frac{2}{5}$ (f) $f(x) = \frac{x^8}{8} - \frac{x^6}{6} + \frac{x^4}{4} - 2$

   (g) $f(x) = \frac{2}{5}x^{\frac{2}{3}} - x^{\frac{-4}{5}} + \frac{3}{x^2}$ (h) $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$

3. (a) If $f(x) = 16x + 2$, find $f'(0), f'(3), f'(8)$

   (b) If $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + x - 16$, find $f'(-1), f'(0), f'(1)$

   (c) If $f(x) = \frac{x^4}{4} + \frac{3}{7}x^7 + 2x - 5$, find $f'(-2)$

   (d) Given that $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$ and hence $\left.\frac{dV}{dr}\right]_{r=2}$

26.8 DERIVATIVE OF PRODUCT OF FUNCTIONS

You are all familiar with the four fundamental operations of Arithmetic : addition, subtraction, multiplication and division. Having dealt with the sum and the difference rules, we now consider the derivative of product of two functions.

Consider $$y = (x^2 + 1)^2$$

This is same as $$y = (x^2 + 1)(x^2 + 1)$$

So we need now to derive the way to find the derivative in such situation.

We write $$y = (x^2 + 1)(x^2 + 1)$$

Let δx be the increment in x and δy the correrponding increment in y. Then

$$y + \delta y = [(x + \delta x)^2 + 1][(x + \delta x)^2 + 1)]$$

$$\Rightarrow \quad \delta y = [(x + \delta x)^2 + 1][(x + \delta x)^2 + 1)] - (x^2 + 1)(x^2 + 1)$$

$$= [(x + \delta x)^2 + 1][(x + \delta x)^2 - x^2)] + (x^2 + 1)[(x + \delta x)^2 + 1] - (x^2 + 1)(x^2 + 1)$$

$$=[(x+\delta x)^2+1][(x+\delta x)^2-x^2]+(x^2+1)[x+\delta x)^2+1-(x^2+1)]$$

$$=[(x+\delta x)^2+1][(x+\delta x)^2-x^2]+(x^2+1)[(x+\delta x)^2-x^2]$$

$\therefore$
$$\frac{\delta y}{\delta x}=\left[(x+\delta x)^2+1\right]\cdot\left[\frac{(x+\delta x)^2-x^2}{\delta x}\right]+\left(x^2+1\right)\left[\frac{(x+\delta x)^2-x^2}{\delta x}\right]$$

$$=\left[(x+\delta x)^2+1\right]\cdot\left[\frac{2x\delta x+(\delta x)^2}{\delta x}\right]+\left(x^2+1\right)\left[\frac{2x\delta x+(\delta x)^2}{\delta x}\right]$$

$$=[(x+\delta x)^2+1](2x+\delta x)+(x^2+1)(2x+\delta x)$$

$\therefore$
$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}[(x+\delta x)^2+1]\cdot[2x+\delta x]+\lim_{\delta x\to 0}(x^2+1)(2x+\delta x)$$

or
$$\frac{dy}{dx}=\left(x^2+1\right)(2x)+\left(x^2+1\right)\cdot(2x)$$

$$=4x\left(x^2+1\right)$$

Let us analyse : $$\frac{dy}{dx}=\left(x^2+1\right)\underbrace{(2x)}_{\substack{\text{derivative}\\ \text{of } x^2+1}}+\left(x^2+1\right)\underbrace{(2x)}_{\substack{\text{derivative}\\ \text{of } x^2+1}}$$

Consider $y=x^3\cdot x^2$

Is $\dfrac{dy}{dx}=x^3\cdot(2x)+x^2\cdot\left(3x^2\right)$?

Let us check $x^3(2x)+x^2\left(3x^2\right)$

$$=2x^4+3x^4$$

$$=5x^4$$

We have $y=x^3\cdot x^2$

$$=x^5$$

$\therefore$ $\dfrac{dy}{dx}=5x^4$

In general, if f (x) and g (x) are two functions of x then the derivative of their product is defined by

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$$

$$=\left[\text{Ist function}\right]\left[\frac{d}{dx}(\text{Second function})\right]+\left[\text{Second function}\right]\left[\frac{d}{dx}(\text{Ist function})\right]$$

which is read as derivative of product of two functions is equal to

= [Ist function] [Derivative of Second function] +

[Second function] [Derivative of Ist function]

This is called the **PRODUCT RULE.**

Example 26.8 Find $\frac{dy}{dx}$, if $y = 5x^6\left(7x^2 + 4x\right)$

Method I. Here y is a product of two functions.

$$\therefore \quad \frac{dy}{dx} = \left(5x^6\right)\cdot\frac{d}{dx}\left(7x^2 + 4x\right) + \left(7x^2 + 4x\right)\frac{d}{dx}\left(5x^6\right)$$

$$= \left(5x^6\right)\left(14x + 4\right) + \left(7x^2 + 4x\right)\left(30x^5\right)$$

$$= 70x^7 + 20x^6 + 210x^7 + 120x^6$$

$$= 280x^7 + 140x^6$$

Method II

$$y = 5x^6\left(7x^2 + 4x\right)$$

$$= 35x^8 + 20x^7$$

$$\therefore \quad \frac{dy}{dx} = 35 \times 8x^7 + 20 \times 7x^6 = 280x^7 + 140x^6$$

which is the same as in Method I.

This rule can be extended to find the derivative of two or more than two functions.

Remark : If f (x), g (x) and h(x) are three given functions of x, then

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right] = f(x)g(x)\frac{d}{dx}h(x) + g(x)h(x)\frac{d}{dx}f(x) + h(x)f(x)\frac{d}{dx}g(x)$$

Example 26.9 Find the derivative of $\left[f(x)g(x)h(x)\right]$ if

$$f(x) = x,\ g(x) = (x-3),\ \text{and} \quad h(x) = x^2 + x$$

Solution : Let $y = x(x-3)\left(x^2 + x\right)$

To find the derivative of y, we can combine any two functions, given on the R.H.S. and apply the product rule or use result mentioned in the above remark.

In other words, we can write

$$y = \left[x(x-3)\right]\left(x^2 + x\right)$$

Let $\quad u(x) = f(x)g(x) = x(x-3) = x^2 - 3x$

Also $\quad h(x) = x^2 + x$

$\therefore \quad y = u(x) \times h(x)$

Hence $\quad \frac{dy}{dx} = x(x-3)\frac{d}{dx}\left(x^2 + x\right) + \left(x^2 + x\right)\frac{d}{dx}\left(x^2 - 3x\right)$

$$= x(x-3)(2x+1) + (x^2+x)(2x-3)$$

$$= x(x-3)(2x+1) + (x^2+x)(x-3) + x(x^2+x)$$

$$= [f(x)g(x)]\cdot h'(x) + [g(x)h(x)]f'(x) + [h(x)f(x)].g'(x)$$

Hence $\qquad \dfrac{d}{dx}[f(x)g(x)h(x)] = [f(x)g(x)]\cdot\dfrac{d}{dx}[h(x)]$

$$+[g(x)h(x)]\frac{d}{dx}[f(x)] + h(x)f(x)\frac{d}{dx}[g(x)]$$

Alternatively, we can directly find the derivative of product of the given three functions.

$$\frac{dy}{dx} = [x(x-3)]\frac{d}{dx}(x^2+x) + [(x-3)(x^2+x)]\frac{d}{dx}(x) + [(x^2+x)\cdot x]\frac{d}{dx}(x-3)$$

$$= x(x-3)(2x+1) + (x-3)(x^2+x)\cdot 1 + (x^2+x)\cdot x\cdot 1$$

$$= 4x^3 - 6x^2 - 6x$$

CHECK YOUR PROGRESS 26.4

1. Find the derivative of each of the following functions by product rule :

(a) $f(x) = (3x+1)(2x-7)$ (b) $f(x) = (x+1)(-3x-2)$

(c) $f(x) = (x+1)(-2x-9)$ (d) $y = (x-1)(x-2)$

(e) $y = x^2(2x^2+3x+8)$ (f) $y = (2x+3)(5x^2-7x+1)$

(g) $u(x) = (x^2-4x+5)(x^3-2)$

2. Find the derivative of each of the functions given below :

(a) $f(r) = r(1-r)(\pi r^2 + r)$ (b) $f(x) = (x-1)(x-2)(x-3)$

(c) $f(x) = (x^2+2)(x^3-3x^2+4)(x^4-1)$

(d) $f(x) = (3x^2+7)(5x-1)(3x^2+9x+8)$

26.9 QUOTIENT RULE

You have learnt sum Rule, Difference Rule and Product Rule to find derivative of a function expressed respectively as either the sum or difference or product of two functions. Let us now take a step further and learn the "Quotient Rule for finding derivative of a function which is the quotient of two functions.

Let $\qquad g(x) = \dfrac{1}{r(x)}, \qquad [r(x) \neq 0]$

Notes

Let us find the derivative of g (x) by first principles

$$g(x)=\frac{1}{r(x)}$$

$$\therefore \qquad g'(x)=\lim_{\delta x\to 0}\left[\frac{\frac{1}{r(x+\delta x)}-\frac{1}{r(x)}}{\delta x}\right]$$

$$=\lim_{\delta x\to 0}\left[\frac{r(x)-r(x+\delta x)}{\delta(x)r(x)r(x+\delta x)}\right]$$

$$=\lim_{\delta x\to 0}\left[\frac{r(x)-r(x+\delta x)}{\delta x}\right]\lim_{\delta x\to 0}\frac{1}{r(x).r(x+\delta x)}$$

$$=-r'(x)\cdot\frac{1}{[r(x)]^2}=-\frac{r'(x)}{[r(x)]^2}$$

Consider any two functions f (x) and g (x) such that $\phi(x)=\frac{f(x)}{g(x)}$, $\quad g(x)\neq 0$

We can write $\phi(x)=f(x)\cdot\frac{1}{g(x)}$

$$\therefore \qquad \phi(x)=f'(x)\cdot\frac{1}{g(x)}+f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right]$$

$$=\frac{f'(x)}{g(x)}+f(x)\left[\frac{-g'(x)}{[g(x)]^2}\right]$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

$$=\frac{\text{(Denominator)(Derivative of Numerator)} - \text{(Numerator)(Derivative of Denominator)}}{\text{(Denominator)}^2}$$

Hence $\qquad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

This is called the quotient Rule.

Example 26.10 Find $f'(x)$ if $f(x)=\frac{4x+3}{2x-1}$, $\quad x\neq\frac{1}{2}$

Solution :

$$f'(x)=\frac{(2x-1)\frac{d}{dx}(4x+3)-(4x+3)\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$= \frac{(2x-1).4-(4x+3).2}{(2x-1)^2}$$

$$= \frac{-10}{(2x-1)^2}$$

Let us consider the following example:

Let $\quad f(x) = \dfrac{1}{2x-1}, \qquad x \neq \dfrac{1}{2}$

$$\frac{d}{dx}\left[\frac{1}{2x-1}\right] = \frac{(2x-1)\dfrac{d}{dx}(1) - 1\dfrac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$= \frac{(2x-1)\times 0 - 2}{(2x-1)^2} \qquad \left[\because \frac{d}{dx}(1) = 0\right]$$

i.e. $\quad \dfrac{d}{dx}\left[\dfrac{1}{2x-1}\right] = -\dfrac{2}{(2x-1)^2}$

Notes

CHECK YOUR PROGRESS 26.5

1. Find the derivative of each of the following :

(a) $y = \dfrac{2}{5x-7}, \; x \neq \dfrac{7}{5}$ (b) $y = \dfrac{3x-2}{x^2+x-1}$ (c) $y = \dfrac{x^2-1}{x^2+1}$

(d) $f(x) = \dfrac{x^4}{x^2-3}$ (e) $f(x) = \dfrac{x^5-2x}{x^7}$ (f) $f(x) = \dfrac{x}{x^2+x+1}$

(g) $f(x) = \dfrac{\sqrt{x}}{x^3+4}$

2. Find $f'(x)$ if

(a) $f(x) = \dfrac{x(x^2+3)}{x-2}, \qquad [x \neq 2]$

(b) $f(x) = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}, \qquad [x \neq 3, \quad x \neq 4]$

26.10 CHAIN RULE

Earlier, we have come across functions of the type $\sqrt{x^4+8x^2+1}$. This function can not be expressed as a sum, difference, product or a quotient of two functions. Therefore, the techniques developed so far do not help us find the derivative of such a function. Thus, we need to develop a rule to find the derivative of such a function.

Let us write : $y=\sqrt{x^4+8x^2+1}$ or $y=\sqrt{t}$ where $t=x^4+8x^2+1$

That is, y is a function of t and t is a function of x. Thus y is a function of a function. We proceed to find the derivative of a function of a function.

Let δt be the increment in t and δy, the corresponding increment in y.

Then $\delta y \to 0$ as $\delta t \to 0$

$$\therefore \qquad \frac{dy}{dt}=\lim_{\delta t\to 0}\frac{\delta y}{\delta t} \qquad \text{(i)}$$

Similarly t is a function of x.

$\therefore \qquad \delta t \to 0 \qquad$ as $\qquad \delta x \to 0$

$$\therefore \qquad \frac{dt}{dx}=\lim_{\delta x\to 0}\frac{\delta t}{\delta x} \qquad \text{(ii)}$$

Here y is a function of t and t is a function of x. Therefore $\delta y \to 0$ as $\delta x \to 0$

From (i) and (ii), we get

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\left[\lim_{\delta t\to 0}\frac{\delta y}{\delta t}\right]\left[\lim_{\delta x\to 0}\frac{\delta t}{\delta x}\right]$$

$$=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

Thus $$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

This is called the **Chain Rule.**

Example 26.11 If $y=\sqrt{x^4+8x^2+1}$, find $\frac{dy}{dx}$

Solution : We are given that

$$y=\sqrt{x^4+8x^2+1}$$

which we may write as

$$y=\sqrt{t}\text{, where } t=x^4+8x^2+1 \qquad \text{(i)}$$

$$\therefore \qquad \frac{dy}{dt}=\frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx}=4x^3+16x$$

Here $$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{1}{2\sqrt{t}}\cdot(4x^3+16x)$$

$$=\frac{4x^3+16x}{2\sqrt{x^4+8x^2+1}}=\frac{2x^3+8x}{\sqrt{x^4+8x^2+1}} \qquad \text{(Using (i))}$$

Example 26.12 Find the derivative of the function $y = \dfrac{5}{(x^2-3)^7}$

Solution :
$$\frac{dy}{dx} = \frac{d}{dx}\left\{5(x^2-3)^{-7}\right\}$$

$$= 5[(-7)(x^2-3)^{-8}]\cdot\frac{d}{dx}(x^2-3) \qquad \text{(Using chain Rule)}$$

$$= -35(x^2-3)^{-8}\cdot(2x)$$

$$= \frac{-70x}{(x^2-3)^8}$$

Example 26.13 Find $\dfrac{dy}{dx}$ where $y = \dfrac{1}{4}v^4$ and $v = \dfrac{2}{3}x^3 + 5$

Solution : We have $y = \dfrac{1}{4}v^4$ and $v = \dfrac{2}{3}x^3 + 5$

$$\frac{dy}{dv} = \frac{1}{4}(4v^3) = v^3 = \left(\frac{2}{3}x^3 + 5\right)^3 \qquad \text{...(i)}$$

and
$$\frac{dv}{dx} = \frac{2}{3}(3x^2) = 2x^2 \qquad \text{..(ii)}$$

Thus
$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx}$$

$$= \left(\frac{2}{3}x^3 + 5\right)^3(2x^2) \qquad \text{[Using (i) and (ii)]}$$

Remark

We have seen in the previous examples that by using various rules of derivatives we can find derivatives of algebraic functions.

CHECK YOUR PROGRESS 26.6

1. Find the derivative of each of the following functions :

(a) $f(x) = (5x-3)^7$ (b) $f(x) = (3x^2-15)^{35}$

(c) $f(x) = (1-x^2)^{17}$ (d) $f(x) = \dfrac{(3-x)^5}{7}$

(e) $y = \dfrac{1}{x^2+3x+1}$ (f) $y = \sqrt[3]{(x^2+1)^5}$

(g) $y = \frac{1}{\sqrt{7-3x^2}}$ (h) $y = \left[\frac{1}{6}x^6 + \frac{1}{2}x^4 + \frac{1}{16}\right]^5$

(i) $y = (2x^2 + 5x - 3)^{-4}$ (j) $y = x + \sqrt{x^2 + 8}$

2. Find $\frac{dy}{dx}$ if

(a) $y = \frac{3-v}{2+v}, v = \frac{4x}{1-x^2}$ (b) $y = at^2, t = \frac{x}{2a}$

Second Order Derivative : Given y is a function of x, say f (x). If the derivative $\frac{dy}{dx}$ is a derivable function of x, then the derivative of $\frac{dy}{dx}$ is known as the second derivative of $y = f(x)$ with respect to x and is denoted by $\frac{d^2y}{dx^2}$. Other symbols used for the second derivative of y are D^2, f", y", y_2 etc.

Remark

Thus the value of f " at x is given by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(h)}{h}$$

The derivatives of third, fourth, ...orders can be similarly defined.

Thus the second derivative, or second order derivative of y with respect to x is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 26.14 Find the second order derivative of

(i) x^2 (ii) $x^3 + 1$ (iii) $(x^2+1)(x-1)$ (iv) $\frac{x+1}{x-1}$

Solution : (i) Let $y = x^2$, then $\frac{dy}{dx} = 2x$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x) = 2 \cdot \frac{d(x)}{dx}$$

$$= 2.1 = 2$$

$$\therefore \quad \frac{d^2y}{dx^2} = 2$$

(ii) Let $y = x^3 + 1$, then

$$\frac{dy}{dx} = 3x^2 \text{ (by sum rule and derivative of a constant is zero)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 3.2x = 6x$$

$\therefore$ $$\frac{d^2y}{dx^2} = 6x$$

(iii) Let $y = (x^2+1)(x-1)$, then

$$\frac{dy}{dx} = (x^2+1)\frac{d}{dx}(x-1) + (x-1),\frac{d}{dx}(x^2+1)$$

$$= (x^2+1)\cdot 1 + (x-1)\cdot 2x \quad \text{or} \quad \frac{dy}{dx} = x^2 + 1 + 2x^2 - 2x = 3x^2 - 2x + 1$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 - 2x + 1\right) = 6x - 2$$

$\therefore$ $$\frac{d^2y}{dx^2} = 6x - 2$$

(iv) Let $y = \frac{x+1}{x-1}$, then

$$\frac{dy}{dx} = \frac{(x-1)\cdot 1 - (x+1).1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-2}{(x-1)^2}\right] = -2.-2.\frac{1}{(x-1)^3} = \frac{4}{(x-1)^3}$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{4}{(x-1)^3}$$

CHECK YOUR PROGRESS 26.7

Find the derivatives of second order for each of the following functions :

(a) x^3 (b) $x^4 + 3x^3 + 9x^2 + 10x + 1$

(c) $\frac{x^2+1}{x+1}$ (d) $\sqrt{x^2+1}$

LET US SUM UP

- The derivative of a function f (x) with respect to x is defined as

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}, \ \delta x > 0$$

- The derivative of a constant is zero i.e., $\frac{dc}{dx} = 0$, where c a is constant.
- Newton's Power Formula

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

- Geometrically, the derivative $\frac{dy}{dx}$ of the function $y = f(x)$ at point $P(x, y)$ is the slope or gradient of the tangent on the curve represented by $y = f(x)$ at the point P.
- The derivative of y with respect to x is the instantaneous rate of change of y with respect to x.
- If f (x) is a derivable function and c is a constant, then

$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$, where $f'(x)$ denotes the derivative of $f(x)$.

- 'Sum or difference rule' of functions :

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Derivative of the sum or difference of two functions is equal to the sum or diference of their derivatives respectively.

- **Product rule :**

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$= (\text{Ist function})\left(\frac{d}{dx}\text{IInd function}\right) + (\text{IInd function})\left(\frac{d}{dx}\text{Ist function}\right)$$

- **Quotient rule :** If $\phi(x) = \frac{f(x)}{g(x)}, \ g(x) \neq 0$, then

$$\phi'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$= \frac{(\text{Denominator})\left(\frac{d}{dx}(\text{Numerator})\right) - \text{Numerator}\left(\frac{d}{dx}(\text{Denominator})\right)}{(\text{Denominator})^2}$$

- **Chain Rule :** $\frac{d}{dx}\left[f\{g(x)\}\right] = f'[g(x)] \cdot \frac{d}{dx}[g(x)]$

  $=$ derivative of $f(x)$ w.r.t $g(x) \times$ derivative of $g(x)$ w.r.t.x

- The derivative of second order of y w.r.t. to x is $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=MKWBx78L7Qg

http://www.youtube.com/watch?v=IiBC4ngwH6E

http://www.youtube.com/watch?v=1015d63VKh4

http://www.youtube.com/watch?v=Bkkk0RLSEy8

http://www.youtube.com/watch?v=ho87DN9wO70

http://www.youtube.com/watch?v=UXQGzgPf1LE

http://www.youtube.com/watch?v=4bZyfvKazzQ

http://www.bbc.co.uk/education/asguru/maths/12methods/03differentiation/index.shtml

TERMINAL EXERCISE

1. The distance s meters travelled in time t seconds by a car is given by the relation $s = t^2$. Caclulate.

   (a) the rate of change of distance with respect to time t.

   (b) the speed of car at time t = 3 seconds.

2. Given $f(t) = 3 - 4t^2$. Use delta method to find $f'(t), f'\left(\frac{1}{3}\right)$.

3. Find the derivative of $f(x) = x^4$ from the first principles. Hence find

   $$f'(0), f'\left(-\frac{1}{2}\right)$$

4. Find the derivative of the function $\sqrt{2x+1}$ from the first principles.

5. Find the derivatives of each of the following functions by the first principles :

   (a) ax + b, where a and b are constants (b) $2x^2 + 5$

   (c) $x^3 + 3x^2 + 5$ (d) $(x-1)^2$

6. Find the derivative of each of the following functions :

(a) $f(x)=px^4+qx^2+7x-11$ (b) $f(x)=x^3-3x^2+5x-8$

(c) $f(x)=x+\frac{1}{x}$ (d) $f(x)=\frac{x^2-a}{a-2}, a \neq 2$

7. Find the derivative of each of the functions given below by two ways, first by product rule, and then by expanding the product. Verify that the two answers are the same.

(a) $y=\sqrt{x}\left(1+\frac{1}{\sqrt{x}}\right)$ (b) $y=x^{\frac{3}{2}}\left(2+5x+\frac{1}{x}\right)$

8. Find the derivative of the following functions :

(a) $f(x)=\frac{x}{x^2-1}$ (b) $f(x)=\frac{3}{(x-1)^2}+\frac{10}{x^3}$

(c) $f(x)=\frac{1}{(1+x^4)}$ (d) $f(x)=\frac{(x+1)(x-2)}{\sqrt{x}}$

(e) $f(x)=\frac{3x^2+4x-5}{x}$ (f) $f(x)=\frac{x-4}{2\sqrt{x}}$

(g) $f(x)=\frac{(x^3+1)(x-2)}{x^2}$

9. Use chain rule, to find the derivative of each of the functions given below :

(a) $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$ (b) $\sqrt{\frac{1+x}{1-x}}$ (c) $\sqrt[3]{x^2(x^2+3)}$

10. Find the derivatives of second order for each of the following :

(a) $\sqrt{x+1}$ (b) $x \cdot \sqrt{x-1}$

ANSWERS

CHECK YOUR PROGRESS 26.1

1. (a) 3 (b) 8 (c) 6 (d) 31
2. 3640 m/s
3. 21 m/s

CHECK YOUR PROGRESS 26.2

1. (a) 10 (b) 2 (c) 6x (d) 2x+5 (e) $21x^2$

2. (a) $-\dfrac{1}{x^2}$ (b) $-\dfrac{1}{ax^2}$ (c) $1-\dfrac{1}{x^2}$ (d) $\dfrac{-a}{(ax+b)^2}$ (e) $\dfrac{ad-bc}{(cx+d)^2}$ (f) $-\dfrac{1}{(3x+5)^2}$

3. (a) $-\dfrac{1}{2x\sqrt{x}}$ (b) $\dfrac{-a}{2(ax+b)(\sqrt{ax+b})}$ (c) $\dfrac{1}{2\sqrt{x}}\left(1-\dfrac{1}{x}\right)$ (d) $\dfrac{2}{(1-x)^2}$

4. (a) $\dfrac{3}{2\sqrt{x}};\dfrac{3}{2\sqrt{2}}$ (b) $2\pi r$; 4π (c) $2\pi r^2$; 36π

CHECK YOUR PROGRESS 26.3

1. (a) 0 (b) 12 (c) 12

2. (a) $180x^8+5$ (b) $-200x^3-40x$ (c) $12x^2-12x$ (d) $5x^8+3$ (e) $3x^2-6x+3$

   (f) $x^7-x^5+x^3$ (g) $\dfrac{4}{15}x^{\frac{-1}{3}}+\dfrac{4}{5}x^{\frac{-9}{5}}-6x^{-3}$ (h) $\dfrac{1}{2\sqrt{x}}+\dfrac{1}{2x^{\frac{3}{2}}}$

3. (a) 16, 16, 16 (b) 3,1 ,1 (c) 186 (d) $4\pi r^2, 16\pi$

CHECK YOUR PROGRESS 26.4

1. (a) $12x-19$ (b) $-6x-5$ (c) $4x-11$ (d) $2x-3$

   (e) $8x^3+9x^2+16x$ (f) $30x^2+2x-19$ (g) $5x^4-16x^3+15x^2-4x+8$

2. (a) $-4\pi r^3+3(\pi-1)r^2+2r$ (b) $3x^2-12x+11$

   (c) $9x^8-28x^7+14x^6-12x^5-5x^4+44x^3-6x^2+4x$

   (d) $(5x-1)(3x^2+9x+8).6x+5(3x^2+7)(3x^2+9x+8)+(3x^2+7)(5x-1)(6x+9)$

CHECK YOUR PROGRESS 26.5

1. (a) $\dfrac{-10}{(5x-7)^2}$ (b) $\dfrac{-3x^2+4x-1}{(x^2+x+1)^2}$ (c) $\dfrac{4x}{(x^2+1)^2}$

   (d) $\dfrac{2x^5-12x^3}{(x^2-3)^2}$ (e) $\dfrac{-2x^4+12}{x^7}$ (f) $\dfrac{1-x^2}{(x^2+x+1)^2}$ (g) $\dfrac{4-5x^3}{2\sqrt{x}(x^3+4)^2}$

2. (a) $\dfrac{2x^3-6x^2-6}{(x-2)^2}$ (b) $\dfrac{-4x^2+20x-22}{(x-3)^2(x-4)^2}$

CHECK YOUR PROGRESS 26.6

1. (a) $35(5x-6)^6$ (b) $210x(3x^2-15)^{34}$ (c) $-34x(1-x^2)^{16}$

   (d) $\dfrac{-5}{7}(3-x)^4$ (e) $-(2x+3)(x^2+3x+1)^{-2}$ (f) $\dfrac{10x}{3}\left(x^2+1\right)^{\frac{2}{3}}$

   (g) $3x(7-3x^2)^{-3/2}$ (h) $5(x^5+2x^3)\left(\dfrac{x^6}{6}+\dfrac{x^4}{2}+\dfrac{1}{16}\right)^4$

   (i) $-4(4x+5)(2x^2+5x-3)^{-5}$ (j) $1+\dfrac{x}{\sqrt{x^2+8}}$

2. (a) $\dfrac{-5(1+x^2)}{(1+2x-x^2)^2}$ (b) $\dfrac{x}{2a}$

CHECK YOUR PROGRESS 26.7

1. (a) $6x$ (b) $12x^2+18x+18$ (c) $\dfrac{4}{(x+1)^3}$ (d) $\dfrac{1}{(1+x^2)^{3/2}}$

TERMINAL EXERCISE

1. (a) 2 t (b) 6 seconds

2 $-8t, -\dfrac{8}{3}$ 3. $0, \dfrac{-1}{2}$ 4. $\dfrac{1}{\sqrt{2x+1}}$

5. (a) a (b) $4x$ (c) $3x^2+6x$ (d) $2(x-1)$

6. (a) $4px^3+2qx+7$ (b) $3x^2-6x+5$ (c) $1-\dfrac{1}{x^2}$ (d) $\dfrac{2x}{a-2}$

7. (a) $\dfrac{1}{2\sqrt{x}}$ (b) $3\sqrt{x}+\dfrac{25}{2}x\sqrt{x}+\dfrac{1}{2\sqrt{x}}$

8. (a) $\dfrac{-(x^2+1)}{(x^2-1)^2}$ (b) $\dfrac{-6}{(x-1)^3}-\dfrac{30}{x^4}$ (c) $\dfrac{-4x^3}{(1+x^4)^2}$ (d) $\dfrac{3}{2}\sqrt{x}-\dfrac{1}{2\sqrt{x}}+\dfrac{1}{x^{3/2}}$

   (e) $3+\dfrac{5}{x^2}$ (f) $\dfrac{1}{4\sqrt{x}}+\dfrac{1}{x\sqrt{x}}$ (g) $3x^2-2-\dfrac{1}{x^2}+\dfrac{4}{x^3}$

9. (a) $1-\dfrac{1}{x^2}$ (b) $\dfrac{1}{\sqrt{1+x}\cdot(1-x)^{\frac{3}{2}}}$ (c) $\dfrac{4x^3+6x}{3(x^4+3x^2)^{\frac{2}{3}}}$

10. (a) $-\dfrac{1}{4(x+1)^{\frac{3}{2}}}$ (b) $\dfrac{2+x-x^2}{4(x-1)^{\frac{1}{2}}}$

27

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

Trigonometry is the branch of Mathematics that has made itself indispensable for other branches of higher Mathematics may it be calculus, vectors, three dimensional geometry, functions-harmonic and simple and otherwise just can not be processed without encountering trigonometric functions. Further within the specific limit, trigonometric functions give us the inverses as well.

The question now arises: Are all the rules of finding the derivative studied by us so far appliacable to trigonometric functions?

This is what we propose to explore in this lesson and in the process, develop the fornulae or results for finding the derivatives of trigonometric functions and their inverses. In all discussions involving the trignometric functions and their inverses, radian measure is used, unless otherwise specifically mentioned.

OBJECTIVES

After studying this lesson, you will be able to:

- find the derivative of trigonometric functions from first principle;
- find the derivative of inverse trigomometric functions from first principle;
- apply product, quotient and chain rule in finding derivatives of trigonometric and inverse trigonometric functions; and
- find second order derivative of a functions.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of trigonometric ratios as functions of angles.
- Standard limits of trigonometric functions
- Definition of derivative, and rules of finding derivatives of function.

27.1 DERIVATIVE OF TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

(i) Let $y = \sin x$

For a small increment δ x in x, let the corresponding increment in y be δ y.

$$\therefore \qquad y+\delta y=\sin\left(x+\delta x\right)$$

and
$$\delta y=\sin\left(x+\delta x\right)-\sin x$$

$$=2\cos\left[x+\frac{\delta x}{2}\right]\sin\frac{\delta x}{2} \qquad \left[\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C+D}{2}\right]$$

$$\therefore \qquad \frac{\delta y}{\delta x}=2\cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\cos\left(x+\frac{\delta x}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=\cos x.1 \qquad \left[\therefore \lim_{\delta x\to 0}\frac{\frac{\sin\delta x}{2}}{\frac{\delta x}{2}}=1\right]$$

Thus
$$\frac{dy}{dx}=\cos x$$

i.e.,
$$\frac{d}{dx}(\sin x)=\cos x$$

(ii) Let $y=\cos x$

For a small increment δ x, let the corresponding increment in y be δ y.

$$\therefore \qquad y+\delta y=\cos\left(x+\delta x\right)$$

and
$$\delta y=\cos\left(x+\delta x\right)-\cos x$$

$$=-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\therefore \qquad \frac{\delta y}{\delta x}=-2\sin\left(x+\frac{\delta x}{2}\right)\cdot\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\lim_{\delta x\to 0}\sin\left(x+\frac{dx}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$=-\sin x\cdot 1$$

Thus, $$\frac{dy}{dx} = -\sin x$$

i.e, $$\frac{d}{dx}(\cos x) = -\sin x$$

(iii) Let y = tan x

For a small increament δx in x, let the corresponding increament in y be δy.

$$\therefore \quad y + \delta y = \tan(x + \delta x)$$

and $$\delta y = \tan(x + \delta x) - \tan x = \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \delta x) \cdot \cos x - \sin x \cdot \cos(x + \delta x)}{\cos(x + \delta x) \cos x} = \frac{\sin[(x + \delta x) - x]}{\cos(x + \delta x) \cos x}$$

$$= \frac{\sin \delta x}{\cos(x + \delta x) \cdot \cos x}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos(x + \delta x) \cos x}$$

or $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdot \lim_{\delta x \to 0} \frac{1}{\cos(x + \delta x) \cos x}$$

$$= 1 \cdot \frac{1}{\cos^2 x} = \sec^2 x \qquad \left[\therefore \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1\right]$$

Thus, $$\frac{dy}{dx} = \sec^2 x$$

i.e. $$\frac{d}{dx}(\tan x) = \sec^2 x$$

(iv) Let y = sec x

For a small increament δ x in, let the corresponding increament in y be δ y .

$$\therefore \quad y + \delta y = \sec(x + \delta x)$$

and $$\delta y = \sec(x + \delta x) - \sec x = \frac{1}{\cos(x + \delta x)} - \frac{1}{\cos x}$$

$$= \frac{\cos x - \cos(x+\delta x)}{\cos(x+\delta x)\cos x} = \frac{2\sin\left[x+\frac{\delta x}{2}\right]\sin\frac{\delta x}{2}}{\cos(x+\delta x)\cos x}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\lim_{\delta x \to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= \frac{\sin x}{\cos^2 x}\cdot 1 = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x.\sec x$$

Thus, $\dfrac{dy}{dx} = \sec x.\tan x$

i.e. $\dfrac{d}{dx}(\sec x) = \sec x \cdot \tan x$

Similarly, we can show that

$$\frac{d}{dx}(\cot x) = -\cos ec^2 x$$

and $\dfrac{d}{dx}(\cos ec x) = -\cos ec\, x \cdot \cot x$

Example 27.1 Find the derivative of $\cot x^2$ from first principle.

Solution: $y = \cot x^2$

For a small increament δx in x, let the corresponding increament in y be δy .

$$\therefore \quad y + \delta y = \cot(x+\delta x)^2$$

$$\delta y = \cot(x+\delta x)^2 - \cot x^2$$

$$= \frac{\cos(x+\delta x)^2}{\sin(x+\delta x)^2} - \frac{\cos x^2}{\sin x^2} = \frac{\cos(x+\delta x)^2\sin x^2 - \cos x^2\sin(x+\delta x)^2}{\sin(x+\delta x)^2\sin x^2}$$

$$= \frac{\sin\left[x^2 - (x+\delta x)^2\right]}{\sin(x+\delta x)^2 \sin x^2} = \frac{\sin\left[-2x\delta x - (\delta x)^2\right]}{\sin(x+\delta x)^2 \sin x^2} = \frac{-\sin\left[(2x+\delta x)\delta x\right]}{\sin(x+\delta x)^2 \sin x^2}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{-\sin\left[(2x+\delta x)\delta x\right]}{\delta x \sin(x+\delta x)^2 \sin x^2}$$

and $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} \frac{\sin\left[(2x+\delta x)\delta x\right]}{\delta x(2x+\delta x)} \lim_{\delta x \to 0} \frac{2x+\delta x}{\sin(x+\delta x)^2 \sin x^2}$$

or $$\frac{dy}{dx} = -1 \cdot \frac{2x}{\sin x^2 . \sin x^2} \qquad \left[\lim_{\delta x \to 0} \frac{\sin\left[(2x+\delta x)\delta x\right]}{\delta x(2x+\delta x)} = 1\right]$$

$$= \frac{-2x}{\left(\sin x^2\right)^2} = \frac{-2x}{\sin^2 x^2} = -2x.\operatorname{cosec}^2 x^2$$

Hence $$\frac{d}{dx}\left(\cot x^2\right) = -2x \cdot \operatorname{cosec}^2 x^2$$

Example 27.2 Find the derivative of $\sqrt{\operatorname{cosec} x}$ from first principle.

Solution: Let $y = \sqrt{\operatorname{cosec} x}$

and $$y + \delta y = \sqrt{\operatorname{cosec}(x+\delta x)}$$

$$\therefore \quad \delta y = \frac{\left[\sqrt{\operatorname{cosec}(x+\delta x)} - \sqrt{\operatorname{cosec} x}\right]\left[\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right]}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}}$$

$$= \frac{\operatorname{cosec}(x+\delta x) - \operatorname{cosec} x}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}} = \frac{\dfrac{1}{\sin(x+\delta x)} - \dfrac{1}{\sin x}}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}}$$

$$= \frac{\sin x - \sin(x+\delta x)}{\left[\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right]\left[\sin(x+\delta x)\sin x\right]}$$

$$= -\frac{2\cos\left(x + \dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\left(\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right)\left[\sin(x+\delta x)\sin x\right]}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x\to 0}\frac{\cos\left(x+\dfrac{\delta x}{2}\right)}{\sqrt{\cos ec\,(x+\delta x)}+\sqrt{\cos ec\, x]}}\times\frac{\dfrac{\sin\delta x/2}{\delta x/2}}{\left[\sin(x+\delta x).\sin x\right]}$$

or $$\frac{dy}{dx} = \frac{-\cos x}{(2\sqrt{(\cos ec\, x)(\sin x)^2}} = -\frac{1}{2}(\cos ec\, x)^{\frac{1}{2}}(\cos ec\, x\cot x)$$

Thus, $$\frac{d}{dx}\left(\sqrt{\cos ec\, x}\right) = \frac{1}{2}(\cos ec\, x)^{\frac{1}{2}}(\cos ec\, x\cot x)$$

Example 27.3 Find the derivative of $\sec^2 x$ from first principle.

Solution: Let y = $\sec^2 x$

and $$y+\delta y = \sec^2(x+\delta x)$$

then, $$\delta y = \sec^2(x+\delta x)-\sec^2 x = \frac{\cos^2 x-\cos^2(x+\delta x)}{\cos^2(x+\delta x)\cos^2 x}$$

$$= \frac{\sin[(x+\delta x+x]\sin[(x+\delta x-x)]}{\cos^2(x+\delta x)\cos^2 x} = \frac{\sin(2x+\delta x)\sin\delta x}{\cos^2(x+\delta x)\cos^2 x}$$

$$\frac{\delta y}{\delta x} = \frac{\sin(2x+\delta x)\sin\delta x}{\cos^2(x+\delta x)\cos^2 x\delta x}$$

Now, $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin(2x+\delta x)\sin\delta x}{\cos^2(x+\delta x)\cos^2 x\delta x}$$

$$\frac{dy}{dx} = \frac{\sin 2x}{\cos^2 x\cos^2 x} = \frac{2\sin x\cos x}{\cos^2 x\cos^2 x} = 2\tan x.\sec^2 x$$

$$= 2\sec x(\sec x.\tan x) = 2\sec x(\sec x\tan x)$$

CHECK YOUR PROGRESS 27.1

1. Find derivative from principle of the following functions with respect to x:

   (a) cosec x (b) cot x (c) cos 2x

   (d) cot 2 x (e) $\cos ec^2 x$ (f) $\sqrt{\sin x}$

2. Find the derivative of each of the following functions:

   (a) $2\sin^2 x$ (b) $\cos ec^2 x$ (c) $\tan^2 x$

27.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

You heve learnt how we can find the derivative of a trigonometric function from first principle and also how to deal with these functions as a function of a function as shown in the alternative method. Now we consider some more examples of these derivatives.

Example 27.4 Find the derivative of each of the following functions:

(i) $\sin 2x$ (ii) $\tan\sqrt{x}$ (iii) $\operatorname{cosec}\left(5x^3\right)$

Solution: (i) Let $y = \sin 2x,$

$= \sin t,$ where t = 2 x

$$\frac{dy}{dt} = \cos t \quad \text{and} \quad \frac{dt}{dx} = 2$$

By chain Rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$, we heve

$$\frac{dy}{dx} = \cos t(2) = 2.\cos t = 2\cos 2x$$

Hence, $\dfrac{d}{dx}(\sin 2x) = 2\cos 2x$

(ii) Let $y = \tan\sqrt{x}$

$= \tan t$ where t = $\sqrt{x}$

$$\therefore \quad \frac{dy}{dt} = \sec^2 t \quad \text{and} \quad \frac{dt}{dx} = \frac{1}{2\sqrt{x}}$$

By chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$, we heve

$$\frac{dy}{dx} = \sec^2 t \cdot \frac{1}{2\sqrt{x}} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

Hence, $\dfrac{d}{dx}\left(\tan\sqrt{x}\right) = \dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$

Alternatively: Let $y = \tan\sqrt{x}$

$$\frac{dy}{dx} = \sec^2\sqrt{x}\,\frac{d}{dx}\sqrt{x} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

(iii) Let $y = \operatorname{cosec}\left(5x^3\right)$

$$\therefore \quad \frac{dy}{dx} = -\operatorname{cosec}\left(5x^3\right)\cot\left(5x^3\right)\cdot\frac{d}{dx}\left[5x^3\right]$$

$$= -15x^2 \operatorname{cosec}\left(5x^3\right)\cot\left(5x^3\right)$$

or you may solve it by substituting $t = 5x^3$

Example 27.5 Find the derivative of each of the following functions:

(i) $y = x^4 \sin 2x$ (ii) $y = \dfrac{\sin x}{1+\cos x}$

Solution : $y = x^4 \sin 2x$

(i) $\therefore \quad \dfrac{dy}{dx} = x^4 \dfrac{d}{dx}(\sin 2x) + \sin 2x \dfrac{d}{dx}\left(x^4\right)$ (Using product rule)

$$= x^4\left(2\cos 2x\right) + \sin 2x\left(4x^3\right)$$

$$= 2x^4 \cos 2x + 4x^3 \sin 2x$$

$$= 2x^3\left[x\cos 2x + 2\sin 2x\right]$$

(ii) Let $y = \dfrac{\sin x}{1+\cos x}$

$$\therefore \quad \frac{dy}{dx} = \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{(1+\cos x)} = \frac{1}{2\cos^2\frac{x}{2}} = \frac{1}{2}\sec^2\frac{x}{2}$$

Example 27.6 Find the derivative of each of the following functions w.r.t. x:

(i) $\cos^2 x$ (ii) $\sqrt{\sin^3 x}$

Solution : (i) Let $y = \cos^2 x$

$= t^2$ where t = cos x

$$\therefore \qquad \frac{dy}{dt} = 2t \text{ and } \frac{dt}{dx} = -\sin x$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}, \text{ we have}$$

$$\frac{dy}{dx} = 2\cos x.(-\sin x)$$

$$= -2\cos x \sin x = -\sin 2x$$

(ii) Let $y = \sqrt{\sin^3 x}$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{2}\left(\sin^3 x\right)^{-1/2} \cdot \frac{d}{dx}\left(\sin^3 x\right) = \frac{1}{2\sqrt{\sin^3 x}} \cdot 3\sin^2 x \cdot \cos x$$

$$= \frac{3}{2}\sqrt{\sin x}\cos x$$

Thus, $$\frac{d}{dx}\left(\sqrt{\sin^3 x}\right) = \frac{3}{2}\sqrt{\sin x}\cos x$$

Example 8.7 Find $\frac{dy}{dx}$, *when*

(i) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$

Solution : We have,

(i) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1-\sin x}{1+\sin x}\right]^{\frac{1}{2}} \cdot \frac{d}{dx}\left[\frac{1-\sin x}{1+\sin x}\right]$$

$$= \frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}} \cdot \frac{(-\cos x)(1+\sin x) - (1-\sin x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}} \cdot \left(\frac{-2\cos x}{(1+\sin x)^2}\right) = \sqrt{\frac{1+\sin x}{1-\sin x} \cdot \frac{\sqrt{1-\sin^2 x}}{(1+\sin x)^2}}$$

$$= -\frac{\sqrt{1+\sin x}\sqrt{1+\sin x}}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

Thus, $dy/dx = -\dfrac{1}{1+\sin x}$

Example 27.8 Find the derivative of each of the following functions at the indicated points :

(i) $y = \sin 2x + (2x-5)^2$ at $x = \dfrac{\pi}{2}$

(ii) $y = \cot x + \sec^2 x + 5$ at $x = \pi/6$

Solution :

(i) $y = \sin 2x + (2x-5)^2$

$$\therefore \quad \frac{dy}{dx} = \cos 2x \frac{d}{dx}(2x) + 2(2x-5)\frac{d}{dx}(2x-5)$$

$$= 2\cos 2x + 4(2x-5)$$

At $x = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = 2\cos\pi + 4(\pi-5) = -2 + 4\pi - 20 = 4\pi - 22$

(ii) $y = \cot x + \sec^2 x + 5$

$$\therefore \quad \frac{dy}{dx} = -\operatorname{cosec}^2 x + 2\sec x(\sec x \tan x) = -\operatorname{cosec}^2 x + 2\sec^2 x \tan x$$

At $x = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = -\operatorname{cosec}^2\dfrac{\pi}{6} + 2\sec^2\dfrac{\pi}{6}\tan\dfrac{\pi}{6} = -4 + 2\cdot\dfrac{4}{3}\dfrac{1}{\sqrt{3}} = -4 + \dfrac{8}{3\sqrt{3}}$

Example 27.9 If $\sin y = x \sin(a+y)$, prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Solution : It is given that

$\sin y = x\sin(a+y)$ or $x = \dfrac{\sin y}{\sin(a+y)}$

Differentiating w.r.t. x on both sides of (1) we get

$$1 = \left[\frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}\right]\frac{dy}{dx}$$

or $$1=\left[\frac{\sin(a+y-y)}{\sin^2(a+y)}\right]\frac{dy}{dx}$$

or $$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

Example 27.10 If y $=\sqrt{\sin x+\sqrt{\sin x+....to\inf inity}},$

prove that $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

Solution : We are given that

$$y=\sqrt{\sin x+\sqrt{\sin x+...to\inf inity}}$$

or $y=\sqrt{\sin x+y}$ or $y^2=\sin x+y$

Differentiating with respect to x, we get

$$2y\frac{dy}{dx}=\cos x+\frac{dy}{dx} \quad \text{or} \quad (2y-1)\frac{dy}{dx}=\cos x$$

Thus, $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

CHECK YOUR PROGRESS 27.2

1. Find the derivative of each of the following functions w.r.tx:

(a) y = 3 sin 4 x (b) y = cos 5 x (c) $y=\tan\sqrt{x}$

(d) $y=\sin\sqrt{x}$ (e) $y=\sin x^2$ (f) $y=\sqrt{2}\tan 2x$

(g) $y=\pi\cot 3x$ (h) $y=\sec 10x$ (i) $y=\cos ec2x$

2. Find the derivative of each of the following functions:

(a) $f(x)=\dfrac{\sec x-1}{\sec x+1}$ (b) $f(x)=\dfrac{\sin x+\cos x}{\sin x-\cos x}$

(c) $f(x)=x\sin x$ (d) $f(x)=(1+x^2)\cos x$

(e) $f(x)=x\cos ec\,x$ (f) $f(x)=\sin 2x\cos 3x$

(g) $f(x)=\sqrt{\sin 3x}$

3. Find the derivative of each of the following functions:

(a) $y = \sin^3 x$ (b) $y = \cos^2 x$ (c) $y = \tan^4 x$

(d) $y = \cot^4 x$ (e) $y = \sec^5 x$ (f) $y = \cos^3 x$

(g) $y = \sec\sqrt{x}$ (h) $y = \sqrt{\dfrac{\sec x + \tan x}{\sec - + \tan x}}$

4. Find the derivative of the following functions at the indicated points:

(a) $y = \cos\left(2x + \pi/2\right), x = \dfrac{\pi}{3}$ (b) $y = \dfrac{1+\sin x}{\cos x}, x = \dfrac{\pi}{4}$

5. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots,}}}$ to infinity

Show that $(2y-1)\dfrac{dy}{dx} = \sec^2 x.$

6. If $\cos y = x\cos(a+y)$,

Prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$

27.3 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

We now find derivatives of standard inverse trignometric functions $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x,$ by first principle.

(i) We will show that by first principle the derivative $\sin^{-1} x$ w.r.t.x is given by

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{\left(1-x^2\right)}}$$

Let $y = \sin^{-1} x$. Then x = sin y and so x + δx = sin (y+δy)

As $\delta x \to 0, \delta y \to 0.$

Now, $\delta x = \sin\left(y+\delta\right) - \sin y$

$$\therefore \quad 1 = \frac{\sin\left(y+\delta y\right) - \sin y}{\delta x}$$ [On dividing both sides by δx]

or $$1=\frac{\sin(y+\delta y)-\sin y}{\delta x}\cdot\frac{\delta y}{\delta x}$$

$\therefore$ $$1=\lim_{\delta x\to 0}\frac{\sin(y+\delta y)-\sin y}{\delta x}\cdot\lim_{\delta x\to 0}\frac{\delta y}{\delta x} \qquad [\because \delta y\to 0 \, when \, \delta x\to 0]$$

$$=\left[\lim_{\delta x\to 0}\frac{2\cos\left(y+\frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)}{\delta x}\right]\cdot\frac{dy}{dx}=(\cos y).\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{(1-\sin^2 y)}}=\frac{1}{\sqrt{(1-x^2)}}$$

$\therefore$ $$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{(1-x^2)}}$$

(ii) $$\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{(1-x^2)}}.$$

For proof proceed exactly as in the case of $\sin^{-1}x$.

(iii) Now we show that,

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

Let $y=\tan^{-1}x$. Then x = tan y and so $x+\delta x=\tan(y+\delta y)$

As $\delta x\to 0, also \, \delta y\to 0$

Now, $\delta x=\tan(y+\delta y)-\tan y$

$\therefore$ $$1=\frac{\tan(y+\delta y)-\tan y}{\delta y}.\frac{\delta y}{\delta x}.$$

$\therefore$ $$1=\lim_{\delta x\to 0}\frac{\tan(y+\delta y)-\tan y}{\delta y}.\lim_{\delta x\to 0}\frac{}{\delta x}. \qquad [\because \delta y\to 0 \, when \, \delta x\to 0]$$

$$=\left[\lim_{\delta x\to 0}\left\{\frac{\sin(y+\delta y)}{\cos(y+\delta y)}-\frac{\sin y}{\cos y}\right\}/\delta y\right].\frac{dy}{dx}$$

$$=\frac{dy}{dx}.\lim_{\delta x\to 0}\frac{\sin(y+\delta y)\cos y-\cos(y+\delta y)\sin y}{\delta y.\cos(y+\delta y)\cos y}$$

$$=\frac{dy}{dx}.\lim_{\delta x\to 0}\frac{\sin(y+\delta y-y)}{\delta y.\cos(y+\delta y)\cos y}$$

$$=\frac{dy}{dx}.\lim_{\delta x\to 0}\left[\frac{\sin\delta y}{\delta y}.\frac{1}{\cos(y+\delta y)\cos y}\right]$$

$$=\frac{dy}{dx}.\frac{1}{\cos^2 y}=\frac{dy}{dx}.\sec^2 y$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}.$$

$$\therefore \quad \frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

(iv) $$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{1}{1+x^2}$$

For proof proceed exactly as in the case of $\tan^{-1} x$.

(v) We have by first principle $\frac{d}{dx}\left(\sec^{-1}x.\right)=\frac{1}{x\sqrt{\left(x^2+1\right)}}$

Let $y=\sec^{-1}x$. Then = sec y and so $x+\delta x=\sec(y+\delta y)$.

As $\delta x\to 0. also\, \delta y\to 0.$

Now $\delta x=\sec(y+\delta y)-\sec y.$

$$\therefore \quad 1=\frac{\sec(y+\delta y)-\sec y}{\delta y}.\frac{\delta y}{\delta x}.$$

$$1=\lim_{\delta x\to 0}\frac{\sec(y+\delta y)-\sec y}{\delta y}.\lim_{\delta x\to 0}\frac{\delta y}{\delta x}. \qquad [\because \delta y\to 0\; when\; \delta x\to 0]$$

$$=\frac{dy}{dx}.\lim_{\delta x\to 0}\frac{2\sin\left(y+\frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)}{\delta y.\cos y\cos\left(y+\delta y\right)}$$

$$=\frac{dy}{dx}.\lim_{\delta x\to 0}\left[\frac{\sin\left(y+\frac{1}{2}\delta y\right)}{\cos y\cos\left(y+\delta y\right)}.\frac{\sin\left(\frac{1}{2}\delta y\right)}{\frac{1}{2}\delta y}\right]$$

$$=\frac{dy}{dx}.\frac{\sin y}{\cos y\cos y}=\frac{dy}{dx}.\sec y\tan y$$

$$\therefore\qquad \frac{dy}{dx}=\frac{1}{\sec y\tan y}=\frac{1}{\sec\sqrt{\left(\sec^2 y-1\right)}}=\frac{1}{x\sqrt{\left(x^2-1\right)}}$$

$$\therefore\qquad \frac{d}{dx}=\left(\sec^{-1}x\right)=\frac{1}{x\sqrt{x^2-1}}$$

(v) $$\frac{d}{dx}=\left(co\sec^{-1}x\right)=\frac{1}{x\sqrt{\left(x^2-1\right)}}.$$

For proof proceed as in the case of $\sec^{-1}x$.

Example 27.11 Find derivative of $\sin^{-1}\left(x^2\right)$ from first principle.

Solution: Let $y=\sin^{-1}x^2$

$$\therefore\qquad x^2=\sin y$$

Now, $$\left(x+\delta x\right)^2=\sin\left(y+\delta y\right)$$

$$\frac{\left(x+\delta x\right)^2-x^2}{\delta x}=\frac{\sin\left(y+\delta x\right)-\sin y}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\left(x+\delta x\right)^2-x^2}{\left(x+\delta x\right)-x}=\lim_{\delta x\to 0}\frac{2\cos\left(y+\frac{\delta x}{2}\right)\sin\frac{\delta y}{2}}{2\cdot\frac{\delta y}{2}}.\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$\Rightarrow \qquad 2x = \cos y . \frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2x}{\cos y} = \frac{2x}{\sqrt{1-\sin^2 y}} = \frac{2x}{\sqrt{1-x^4}}.$$

Example 27.12 Find derivative of $\sin^{-1}\sqrt{x}$ w.r.t.x by first principle

Solution: Let $\quad y = \sin^{-1}\sqrt{x}$

$$\Rightarrow \qquad \sin y = \sqrt{x}$$

Also $\quad \sin(y+\delta y) = \sqrt{x+\delta x}$

From (1) and (2), we get

$$\sin(y+\delta y) - \sin y = \sqrt{x+\delta x} - \sqrt{x}$$

or $\quad 2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right) = \frac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)\left(\sqrt{x+\delta x}+\sqrt{x}\right)}{\sqrt{x+\delta x}+\sqrt{x}}$

$$= \frac{\delta x}{\sqrt{x+\delta x}+\sqrt{x}}$$

$\therefore \quad \frac{2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)}{\delta x} = -\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$

or $\quad \frac{\delta y}{\delta x}.\cos\left(y+\frac{\delta y}{2}\right).\frac{\sin\left(\frac{\delta y}{2}\right)}{\frac{\delta y}{2}} = \frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$

$\therefore \quad \lim_{\delta x\to 0}\frac{\delta y}{\delta x}.\lim_{\delta x\to 0}\cos\left(y+\frac{\delta y}{2}\right).\lim_{\delta x\to 0}\frac{\sin\left(\frac{\delta y}{2}\right)}{\frac{\delta y}{2}}$

$$= \lim_{\delta x\to 0}\frac{1}{\sqrt{x+\delta x}+\sqrt{x}} \qquad (\because \delta y \to 0 \; as \; \delta x \to 0)$$

or $\quad \frac{dy}{dx}\cos = \frac{1}{2\sqrt{x}}$ or $\frac{dy}{dx} = \frac{1}{2\sqrt{x}\cos y} = \frac{1}{2\sqrt{x}\sqrt{1-\sin^2 y}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

CHECK YOUR PROGRESS 27.3

1. Find by first principle that derivative of each of the following:

(i) $\cos^{-1} x^2$ (ii) $\frac{\cos^{-1} x}{x}$ (iii) $\cos^{-1}\sqrt{x}$

(iv) $\tan^{-1} x^2$ (v) $\frac{\tan^{-1} x}{x}$ (vi) $\tan^{-1}\sqrt{x}$

27.4 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

In the previous section, we have learnt to find derivatives of inverse trignometric functions by first principle. Now we learn to find derivatives of inverse trigonometric functions using these results

Example 27.13 Find the derivative of each of the following:

(i) $\sin^{-1}\sqrt{x}$ (ii) $\cos^{-1} x^2$ (iii) $\left(\cos^{-1} x\right)^2$

Solution:

(i) Let $y = \sin^{-1}\sqrt{x}$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{\sqrt{1-x}}.\frac{1}{2}\times^{-1/2} = \frac{1}{2\sqrt{\times}\sqrt{1-x}}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{2\sqrt{\times}\sqrt{1-x}}$$

(iii) Let $y = \cos^{-1} x^2$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(x^2\right)^2}}.\frac{d}{dx}\left(x^2\right) = \frac{-1}{\sqrt{1-x^4}}.(2x)$$

$$\therefore \qquad \frac{d}{dx}\left(\cos^{-1} x^2\right) = \frac{-2x}{\sqrt{1-x^4}}$$

(iii) Let $y = \left(\cos^{-1} x\right)^2$

$$\frac{dy}{dx} = 2\left(\cos ec^{-1}x\right).\frac{d}{dx}\left(\cos ec^{-1}x\right) = 2\left(\cos ec^{-1}x\right).\frac{1}{|x|\sqrt{x^2-1}}$$

$$= \frac{-2\cos ec^{-1}x}{|x|\sqrt{x^2-1}}$$

$$\therefore \quad \frac{d}{dx}\left(\cos ec^{-1}x\right)^2 = \frac{-2\cos ec^{-1}x}{|x|\sqrt{x^2-1}}$$

Example 27.14 Find the derivative of each of the following:

(i) $\tan^{-1}\dfrac{\cos x}{1+\sin x}$ (ii) $\sin\left(2\sin^{-1}x\right)$

Solution:

(i) Let $y = \tan^{-1}\dfrac{\cos x}{1+\sin x} = \tan^{-1}\dfrac{\sin\left(\dfrac{\pi}{2}-2\right)}{1+\cos\left(\dfrac{\pi}{2}-x\right)}$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right] = \tan\frac{\pi}{4}-\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = -1/2$$

(ii) $y = \sin\left(2\sin^{-1}x\right)$

Let $y = \sin\left(2\sin^{-1}x\right)$

$$\therefore \quad \frac{dy}{dx} = \cos\left(2\sin^{-1}x\right).\frac{d}{dx}\left(2\sin^{-1}x\right)$$

$$\therefore \quad \frac{dy}{dx} = \cos\left(2\sin^{-1}x\right).\frac{2}{\sqrt{1-x^2}}$$

$$= \frac{2\cos\left(2\sin^{-1}x\right)}{\sqrt{1-x^2}}$$

Example 27.15 Show that the derivative of $\tan^{-1}\dfrac{2x}{1-x^2}$ $w.r.t \sin^{-1}\dfrac{2x}{1+x^2}$ is 1.

Solution: Let $y = \tan^{-1}\dfrac{2x}{1-x^2}$ and $z = \sin^{-1}\dfrac{2x}{1+x^2}$

Let $x = \tan\theta$

$$y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} \textit{ and } z = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \tan^{-1}(\tan 2\theta) \textit{ and } z = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta \qquad \textit{and } z = 2\theta$$

$$\frac{dy}{d\theta} - 2 \qquad \text{and} \quad \frac{dz}{d\theta} = 2$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dz} = 2\cdot\frac{1}{2} = 1 \qquad \text{(By chain rule)}$$

CHECK YOUR PROGRESS 27.4

Find the derivative of each of the following functions w.r.t. x and express the result in the simplest form (1-3):

1. (a) $\sin^{-1} x^2$ (b) $\cos^{-1}\frac{x}{2}$ (c) $\cos^{-1}\frac{1}{x}$

2. (a) $\tan^{-1}(\cos ec\, x - \cot x)$ (b) $\cot^{-1}(\sec x + \tan x)$ (c) $\cot^{-1}\frac{\cos x - \sin x}{\cos x + \sin x}$

3. (a) $\sin(\cos^{-1} x)$ (b) $\sec(\tan^{-1} x)$ (c) $\sin^{-1}(1-2x^2)$

   (d) $\cos^{-1}(4x^3 - 3x)$ (e) $\cot^{-1}\left(\sqrt{1+x^2} + x\right)$

4. Find the derivative of:

$$\frac{\tan^{-1} x}{1+\tan^{-1} x} \; w.r.t \tan^{-1} x.$$

27.5 SECOND ORDER DERIVATIVES

We know that the second order derivative of a functions is the derivative of the first derivative of that function. In this section, we shall find the second order derivatives of trigonometric and inverse trigonometric functions. In the process, we shall be using product rule, quotient rule and chain rule.

Let us take some examples.

Example 27.16 Find the second order derivative of

(i) sin x (ii) x cos x (iii) $\cos^{-1} x$

Solution: (i) Let $y = \sin x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \cos x$$

Differentiating w.r.t. x both sides again, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos x) = -\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\sin x$$

(ii) Let $y = x \cos x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = x(-\sin x) + \cos.1$$

$$\frac{dy}{dx} = -x\sin x + \cos x$$

Differentiating w.r.t. x both sides again, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-x\sin x + \cos x) = -(x.\cos x + \sin x) - \sin x$$

$$= -x.\cos x - 2\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -(x.\cos x + 2\sin x)$$

(iii) Let $y = \cos^{-1} x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\left(1-x^2\right)^{1/2}} = -\left(1-x^2\right)^{\frac{1}{2}}$$

Differentiating w.r.t. x both sides, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{-1}{2}\cdot\left(1-x^2\right)^{-3/2}\cdot(-2x)\right] = -\frac{x}{\left(1-x^2\right)^{-3/2}}$$

$$\frac{d^2y}{dx^2} = \frac{-x}{\left(1-x^2\right)^{-3/2}}$$

Example 27.17 If $y=\sin^{-1}x$, show that $\left(1-x^2\right)y_2-xy_1=0$, where y_2 and y_1 respectively denote the second and first, order derivatives of y w.r.t. x.

Solution: We have, $y=\sin^{-1}x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

or $$\left(\frac{dy}{dx}\right)^2=\frac{1}{1-x^2} \qquad \text{(Squaring both sides)}$$

or $$\left(1-x^2\right)\left(y_1\right)^2=1$$

Differentiating w.r.t. x both sides, we get

$$\left(1-x^2\right)\cdot 2y_1\frac{d}{dx}\left(y_1\right)+\left(-2x\right)\cdot y_1^2=0$$

or $$\left(1-x^2\right)\cdot 2y_1y_2-2xy_1^2=0$$

or $$\left(1-x^2\right)y_2-xy_1=0$$

CHECK YOUR PROGRESS 27.5

1. Find the second order derivative of each of the following:

   (a) $\sin\left(\cos x\right)$ (b) $x^2\tan^{-1}x$

2. If $y=\frac{1}{2}\left(\sin^{-1}x\right)^2$, show that $\left(1-x^2\right)y_2-xy_1=1$.

3. If $y=\sin\left(\sin x\right)$, prove that $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$.

4. If y = x + tan x, show that $\cos^2 x\frac{d^2y}{dx^2}-2y+2x=0$

LET US SUM UP

- (i) $\frac{d}{dx}\left(\sin x\right)=\cos x$ (ii) $\frac{d}{dx}\left(\cos x\right)=-\sin x$

  (iii) $\frac{d}{dx}\left(\tan x\right)=\sec^2 x$ (iv) $\frac{d}{dx}\left(\cot x\right)=-co\sec^2 x$

MODULE - VIII
Calculus

Notes

(v) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (vi) $\frac{d}{dx}(\operatorname{cosec} x) = -co\sec x \cot x$

- If u is a derivable function of x, then

(i) $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ (ii) $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

(iii) $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$ (iv) $\frac{d}{dx}(\cot u) = -\cos ec^2 u \frac{du}{dx}$

(v) $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$ (vi) $\frac{d}{dx}(\cos ecu) = -\cos ecu \cot u \frac{du}{dx}$

- (i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{-1}{1-x^2}$ (iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$ (vi) $\frac{d}{dx}(\cos ec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

- If u is a derivable function of x, then

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$ (ii) $\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

(iii) $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$ (iv) $\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$

(v) $\frac{d}{dx}(\sec^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$ (vi) $\frac{d}{dx}(\cos ec^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

The second order derivative of a trignometric function is the derivative of their first order derivatives.

SUPPORTIVE WEB SITES

http://people.hofstra.edu/stefan_waner/trig/trig3.html

http://www.math.com/tables/derivatives/more/trig.htm

https://www.freemathhelp.com/trig-derivatives.html

TERMINAL EXERCISE

1. If $y = x^3 \tan^2 \dfrac{x}{2}$, *find* $\dfrac{dy}{dx}$.

2. Evaluate, $\dfrac{d}{dx}\sqrt{\sin^4 x + \cos^4 x}$ *at* $x = \dfrac{\pi}{2}$ and 0.

3. If $y = \dfrac{5x}{\sqrt[3]{(1-x)^2}} + \cos^2(2x+1)$, find $\dfrac{dy}{dx}$.

4. $y = \sec^{-1}\dfrac{\sqrt{x+1}}{\sqrt{x-1}} + \sin^{-1}\dfrac{\sqrt{x-1}}{\sqrt{x}}$, then show that $\dfrac{dy}{dx} = 0$

5. If $x = a\cos^3\theta, y = a\sin^3\theta$, then find $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

6. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}$, find $\dfrac{dy}{dx}$.

7. Find the derivative of $\sin^{-1x}$ $w.r.t \cos^{-1}\sqrt{1-x^2}$

8. If y = cos (cos x), prove that

   $$\frac{d^2y}{dx^2} - \cot x \cdot \frac{dy}{dx} + y.\sin^2 x = 0.$$

9. If $y = \tan^{-1} x$ show that

   $$(1+x)^2 y_2 + 2xy_1 = 0.$$

10. If $y = \left(\cos^{-1} x\right)^2$ show that

    $$(1+x)^2 y_2 - xy_1 - 2 = 0.$$

ANSWERS

CHECK YOUR PROGRESS 27.1

(1) (a) $-\cos ec x \cot x$ (b) $-\cos ec^2 x$ (c) -2 sin 2 x

(d) $-2\cos ec^2 2x$ (e) $-2x \cos ec x^2 \cot x^2$ (f) $\dfrac{\cos x}{2\sqrt{\sin x}}$

2. (a) 2 sin 2 x (b) $-2\cos ec^2 x \cot x$ (c) $2\tan x \sec^2 x$

CHECK YOUR PROGRESS 27.2

1. (a) 12 cos 4 x (b) - 5 sin 5 x (c) $\dfrac{\sec^2 \sqrt{x}}{2\sqrt{x}}$

(d) $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$ (e) $2x \cos x^2$ (f) $2\sqrt{2}\sec^2 2x$

(g) $-3\pi \cos ec^2 3x$ (h) 10 sec 10 x tan 10 x

(I) $-2\cos ec 2x \cot 2x$

2. (a) $\dfrac{2\sec x \tan x}{(\sec x+1)^2}$ (b) $\dfrac{-2}{(\sin x-\cos x)^2}$ (c) $x\cos x+\sin x$

(d) $2x\cos x-(1+x^2)\sin x$ (e) $\cos ec x(1-x\cot x)$

(f) $2\cos 2x \cos 3x-3\sin 2x \sin 3x$ (g) $\dfrac{3\cos 3x}{2\sqrt{\sin 3x}}$

3. (a) $3\sin^2 x \cos x$ (b) $-\sin 2x$ (c) $4\tan^3 x \sec^2 x$

(d) $-4\cot^3 x \cos ec^2 x$ (e) $5\sec^5 x \tan x$ (f) $-3\cos ec^3 x \cot x$

(g) $\dfrac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}$ (h) $\sec x(\sec x+\tan x)$

4. (a) 1 (b) $\sqrt{2}+2$

CHECK YOUR PROGRESS 27.3

1. (i) $\dfrac{-2x}{\sqrt{1-x^4}}$ (ii) $\dfrac{-1}{x\sqrt{1-x^2}}-\dfrac{-\cos^{-1}x}{x^2}$

(iii) $\dfrac{-1}{2x^{\frac{1}{2}}\sqrt{(1-x)}}$ (iv) $\dfrac{2x}{1+x^2}$

(v) $\dfrac{1}{x\left(1+x^2\right)}-\dfrac{\tan^{-1}x}{x^2}$ (vi) $\dfrac{-1}{2x^{\frac{1}{2}}\sqrt{(1-x)}}$

CHECK YOUR PROGRESS 27.4

1. (a) $\dfrac{2x}{\sqrt{1-x^4}}$ (b) $\dfrac{-1}{\sqrt{4-x^2}}$

(c) $\dfrac{1}{x\sqrt{x^2-1}}$

2. (a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$

(c) -1

3. (a) $\dfrac{\cos\left(\cos^{-1}x\right)}{\sqrt{1-x^2}}$ (b) $\dfrac{x}{1+x^2}\cdot\sec\left(\tan^{-1}x\right)$

(c) $\dfrac{-2}{\sqrt{1-x^2}}$ (d) $\dfrac{-3}{\sqrt{1-x^2}}$

(e) $\dfrac{-1}{2\left(1+x^2\right)}$

4. $\dfrac{1}{\left(1+\tan^{-1}x\right)^2}$

CHECK YOUR PROGRESS 27.5

1. (a) $-\cos x\cos(\cos x)-\sin^2 x\sin(\cos x)$

(b) $\dfrac{2x\left(2+x^2\right)}{\left(1+x^2\right)^2}+2\tan^{-1}x$

TERMINAL EXERCISE

1. $x^3 \tan\frac{x}{2}\sec^2\frac{x}{2}+3x^2\tan^2\frac{x}{2}$

2. 0,0

3. $\frac{5(3-x)}{3(1-x)^{\frac{5}{3}}}-2\sin(4x+2)$

5. $|\sec\theta|$

6. $\frac{1}{2y-1}$

7. $\frac{1}{2\sqrt{1-x^2}}$

28

DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We are aware that population generally grows but in some cases decay also. There are many other areas where growth and decay are continuous in nature. Examples from the fields of Economics, Agriculture and Business can be cited, where growth and decay are continuous. Let us consider an example of bacteria growth. If there are 10,00,000 bacteria at present and say they are doubled in number after 10 hours, we are interested in knowing as to after how much time these bacteria will be 30,00,000 in number and so on.

Answers to the growth problem does not come from addition (repeated or otherwise), or multiplication by a fixed number. In fact Mathematics has a tool known as exponential function that helps us to find growth and decay in such cases. Exponential function is inverse of logarithmic function. We shall also study about Rolle's Theorem and Mean Value Theorems and their applications. In this lesson, we propose to work with this tool and find the rules governing their derivatives.

OBJECTIVES

After studying this lesson, you will be able to :

- define and find the derivatives of exponential and logarithmic functions;
- find the derivatives of functions expressed as a combination of algebraic, trigonometric, exponential and logarithmic functions; and
- find second order derivative of a function.
- state Rolle's Theorem and Lagrange's Mean Value Theorem; and
- test the validity of the above theorems and apply them to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Application of the following standard limits :

(i) $$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

(ii) $$\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$$

(iii) $$\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}=1$$

- Definition of derivative and rules for finding derivatives of functions.

28.1 DERIVATIVE OF EXPONENTIAL FUNCTIONS

Let $y = e^x$ be an exponential function. .....(i)

$\therefore \quad y + \delta y = e^{(x+\delta x)}$ (Corresponding small increments) .....(ii)

From (i) and (ii), we have

$$\therefore \quad \delta y = e^{x+\delta x} - e^x$$

Dividing both sides by δx and taking the limit as $\delta x \to 0$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} e^x \frac{\left[e^{\delta x} - 1\right]}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x.1 = e^x$$

Thus, we have $\mathbf{\frac{d}{dx}\left(e^x\right) = e^x}$.

Working rule : $\quad \frac{d}{dx}\left(e^x\right) = e^x \cdot \frac{d}{dx}(x) = e^x$

Next, let $\quad y = e^{ax+b}$.

Then $\quad y + \delta y = e^{a(x+\delta x)+b}$

[δx and δy are corresponding small increments]

$$\therefore \quad \delta y = e^{a(x+\delta x)+b} - e^{ax+b}$$

$$= e^{ax+b}\left[e^{a\delta x} - 1\right]$$

$$\therefore \quad \frac{\delta y}{\delta x} = e^{ax+b} \frac{\left[e^{a\delta x} - 1\right]}{\delta x}$$

$$= a \cdot e^{ax+b} \frac{e^{a\delta x} - 1}{a\delta x} \qquad \text{[Multiply and divide by a]}$$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = a \cdot e^{ax+b} \cdot \lim_{\delta x \to 0} \frac{e^{a\delta x} - 1}{a\ \delta x}$$

or

$$\frac{dy}{dx} = a \cdot e^{ax+b} \cdot 1 \qquad \left[\lim_{x \to 0} \frac{e^x - 1}{x} = 1\right] \quad = ae^{ax+b}$$

Working rule :

$$\frac{d}{dx}\left(e^{ax+b}\right)=e^{ax+b}\cdot\frac{d}{dx}(ax+b)=e^{ax+b}\cdot a$$

$$\therefore \quad \frac{d}{dx}\left(e^{ax+b}\right)=ae^{ax+b}$$

Example 28.1 Find the derivative of each of the follwoing functions :

(i) e^{5x} (ii) e^{ax} (iii) $e^{-\frac{3x}{2}}$

Soution : (i) Let $y=e^{5x}$.

Then $y=e^{t}$ where $5x=t$

$$\therefore \quad \frac{dy}{dt}=e^{t} \quad \text{and} \quad 5=\frac{dt}{dx}$$

We know that, $\dfrac{dy}{dx}=\dfrac{dy}{dt}\cdot\dfrac{dt}{dx}=e^{t}\cdot 5=5e^{5x}$

Alternatively $\dfrac{d}{dx}\left(e^{5x}\right)=e^{5x}\cdot\dfrac{d}{dx}(5x)=e^{5x}\cdot 5=5e^{5x}$

(ii) Let $y=e^{ax}$.

Then $y=e^{t}$ when $t=ax$

$$\therefore \quad \frac{dy}{dt}=e^{t} \text{ and } \frac{dt}{dx}=a$$

We know that, $\dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{dx}=e^{t}\cdot a$

Thus, $\dfrac{dy}{dx}=a\cdot e^{ax}$

(iii) Let $y=e^{\frac{-3x}{2}}$

$$\therefore \quad \frac{dy}{dt}=e^{\frac{-3}{2}x}\cdot\frac{d}{dx}\left(\frac{-3}{2}x\right)$$

Thus, $\dfrac{dy}{dt}=\dfrac{-3}{2}e^{\frac{-3x}{2}}$

Example 28.2 Find the derivative of each of the following :

(i) $y=e^{x}+2\cos x$ (ii) $y=e^{x^2}+2\sin x-\dfrac{5}{3}e^{x}+2e$

Solution : (i) $y = e^x + 2\cos x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) + 2\frac{d}{dx}(\cos x) = e^x - 2\sin x$$

(ii) $y = e^{x^2} + 2\sin x - \frac{5}{3}e^x + 2e$

$$\therefore \quad \frac{dy}{dx} = e^{x^2}\frac{d}{dx}\left(x^2\right) + 2\cos x - \frac{5}{3}e^x + 0 \quad \text{[Since e is constant]}$$

$$= 2xe^{x^2} + 2\cos x - \frac{5}{3}e^x$$

Example 28.3 Find $\frac{dy}{dx}$, when

(i) $y = e^{x\cos x}$ (ii) $y = \frac{1}{x}e^x$ (iii) $y = e^{\frac{1-x}{1+x}}$

Solution : (i) $y = e^{x\cos x}$

$$\therefore \quad \frac{dy}{dx} = e^{x\cos x}\frac{d}{dx}(x\cos x)$$

$$\therefore \quad \frac{dy}{dx} = e^{x\cos x}\left[x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}(x)\right]$$

$$= e^{x\cos x}\left[-x\sin x + \cos x\right]$$

(ii) $y = \frac{1}{x}e^x$

$$\therefore \quad \frac{dy}{dx} = e^x\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\frac{d}{dx}\left(e^x\right) \qquad \text{[Using product rule]}$$

$$= \frac{-1}{x^2}e^x + \frac{1}{x}e^x$$

$$= \frac{e^x}{x^2}[-1+x] = \frac{e^x}{x^2}[x-1]$$

(iii) $y = e^{\frac{1-x}{1+x}}$

$$\frac{dy}{dx} = e^{\frac{1-x}{1+x}}\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= e^{\frac{1-x}{1+x}}\left[\frac{-1.(1+x)-(1-x).1}{(1+x)^2}\right]$$

$$= e^{\frac{1-x}{1+x}}\left[\frac{-2}{(1+x)^2}\right] = \frac{-2}{(1+x)^2}e^{\frac{1-x}{1+x}}$$

Example 28.4 Find the derivative of each of the following functions :

(i) $e^{\sin x} \cdot \sin e^x$ (ii) $e^{ax} \cdot \cos(bx+c)$

Solution : $y = e^{\sin x} \cdot \sin e^x$

$$\therefore \quad \frac{dy}{dx} = e^{\sin x} \cdot \frac{d}{dx}\left(\sin e^x\right) + \sin e^x \frac{d}{dx} e^{\sin x}$$

$$= e^{\sin x} \cdot \cos e^x \cdot \frac{d}{dx}\left(e^x\right) + \sin e^x \cdot e^{\sin x} \frac{d}{dx}(\sin x)$$

$$= e^{\sin x} \cdot \cos e^x \cdot e^x + \sin e^x \cdot e^{\sin x} \cdot \cos x$$

$$= e^{\sin x}[e^x \cdot \cos e^x + \sin e^x \cdot \cos x]$$

(ii) $y = e^{ax}\cos(bx+c)$

$$\therefore \quad \frac{dy}{dx} = e^{ax} \cdot \frac{d}{dx}\cos(bx+c) + \cos(bx+c)\frac{d}{dx}e^{ax}$$

$$= e^{ax} \cdot [-\sin(bx+c)]\frac{d}{dx}(bx+c) + \cos(bx+c)e^{ax}\frac{d}{dx}(ax)$$

$$= -e^{ax}\sin(bx+c) \cdot b + \cos(bx+c)e^{ax}.a$$

$$= e^{ax}[-b\sin(bx+c) + a\cos(bx+c)]$$

Example 28.5 Find $\frac{dy}{dx}$, if $y = \frac{e^{ax}}{\sin(bx+c)}$

Solution :

$$\frac{dy}{dx} = \frac{\sin(bx+c)\frac{d}{dx}e^{ax} - e^{ax}\frac{d}{dx}[\sin(bx+c)]}{\sin^2(bx+c)}$$

$$= \frac{\sin(bx+c).e^{ax}.a - e^{ax}\cos(bx+c).b}{\sin^2(bx+c)}$$

$$= \frac{e^{ax}[a\sin(bx+c) - b\cos(bx+c)]}{\sin^2(bx+c)}$$

CHECK YOUR PROGRESS 28.1

1. Find the derivative of each of the following functions :

(a) e^{5x} (b) e^{7x+4} (c) $e^{\sqrt{2}x}$ (d) $e^{\frac{-7}{2}x}$ (e) e^{x^2+2x}

2. Find $\frac{dy}{dx}$, if

(a) $y = \frac{1}{3}e^x - 5e$ (b) $y = \tan x + 2\sin x + 3\cos x - \frac{1}{2}e^x$

(c) $y = 5\sin x - 2e^x$ (d) $y = e^x + e^{-x}$

3. Find the derivative of each of the following functions :

(a) $f(x) = e^{\sqrt{x+1}}$ (b) $f(x) = e^{\sqrt{\cot x}}$

(c) $f(x) = e^{x\sin^2 x}$ (d) $f(x) = e^{x\sec^2 x}$

4. Find the derivative of each of the following functions :

(a) $f(x) = (x-1)e^x$ (b) $f(x) = e^{2x}\sin^2 x$

5. Find $\frac{dy}{dx}$, if

(a) $y = \frac{e^{2x}}{\sqrt{x^2+1}}$ (b) $y = \frac{e^{2x}\cdot\cos x}{x\sin x}$

28.2 DERIVATIVE OF LOGARITHMIC FUNCTIONS

We first consider logarithmic function

Let $y = \log x$ .....(i)

$\therefore$ $y + \delta y = \log(x + \delta x)$ .....(ii)

(δx and δy are corresponding small increments in x and y)

From (i) and (ii), we get

$$\delta y = \log(x+\delta x) - \log x$$

$$= \log\frac{x+\delta x}{x}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{1}{\delta x}\log\left[1+\frac{\delta x}{x}\right]$$

$$= \frac{1}{x}\cdot\frac{x}{\delta x}\log\left[1+\frac{\delta x}{x}\right] \qquad \text{[Multiply and divide by x]}$$

$$= \frac{1}{x}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

Taking limits of both sides, as $\delta x \to 0$, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{1}{x}\lim_{\delta x\to 0}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log\left\{\lim_{\delta x \to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right\}$$

$$= \frac{1}{x}\log e \qquad \left[\because \lim_{\delta x \to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = e\right]$$

$$= \frac{1}{x}$$

Thus, $$\mathbf{\frac{d}{dx}(\log x) = \frac{1}{x}}$$

Next, we consider logarithmic function

$$y = \log(ax+b) \qquad \text{...(i)}$$

$$\therefore \quad y + \delta y = \log[a(x+\delta x)+b] \qquad \text{...(ii)}$$

[δx and δy are corresponding small increments]

From (i) and (ii), we get

$$\delta y = \log[a(x+\delta x)+b] - \log(ax+b)$$

$$= \log\frac{a(x+\delta x)+b}{ax+b}$$

$$= \log\frac{(ax+b)+a\delta x}{ax+b}$$

$$= \log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{1}{\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$= \frac{a}{ax+b} \cdot \frac{ax+b}{a\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right] \qquad \left[\text{Multiply and divide by } \frac{a}{ax+b}\right]$$

$$= \frac{a}{ax+b}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

Taking limits on both sides as $\delta x \to 0$

$$\therefore \quad \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{a}{ax+b}\lim_{\delta x \to 0}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

or $$\frac{dy}{dx} = \frac{a}{ax+b}\log e \qquad \left[\because \lim_{x \to 0}(1+x)^{\frac{1}{x}} = e\right]$$

or, $$\frac{dy}{dx} = \frac{a}{ax+b}$$

Working rule :

$$\frac{d}{dx}\log(ax+b)=\frac{1}{ax+b}\frac{d}{dx}(ax+b)$$

$$=\frac{1}{ax+b}\times a=\frac{a}{ax+b}$$

Example 28.6 Find the derivative of each of the functions given below :

(i) $y=\log x^5$ (ii) $y=\log\sqrt{x}$ (iii) $y=(\log x)^3$

Solution : (i) $y=\log x^5=5\log x$

$\therefore$ $\frac{dy}{dx}=5\cdot\frac{1}{x}=\frac{5}{x}$

(ii) $y=\log\sqrt{x}=\log x^{\frac{1}{2}}$ or $y=\frac{1}{2}\log x$

$\therefore$ $\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{x}=\frac{1}{2x}$

(iii) $y=(\log x)^3$

$\therefore$ $y=t^3,$ when $t=\log x$

$\Rightarrow$ $\frac{dy}{dt}=3t^2$ and $\frac{dt}{dx}=\frac{1}{x}$

We know that, $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=3t^2\cdot\frac{1}{x}$

$\therefore$ $\frac{dy}{dx}=3(\log x)^2\cdot\frac{1}{x}$

$\therefore$ $\frac{dy}{dx}=\frac{3}{x}(\log x)^2$

Example 28.7 Find, $\frac{dy}{dx}$ if

(i) $y=x^3\log x$ (ii) $y=e^x\log x$

Solution :

(i) $y=x^3\log x$

$\therefore$ $\frac{dy}{dx}=\log x\frac{d}{dx}(x^3)+x^3\frac{d}{dx}(\log x)$ [Using Product rule]

$$=3x^2\log x+x^3\cdot\frac{1}{x}$$

$$= x^2(3\log x + 1)$$

(ii) $$y = e^x \log x$$

$$\therefore \quad \frac{dy}{dx} = e^x \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx} e^x$$

$$= e^x \cdot \frac{1}{x} + e^x \cdot \log x$$

$$= e^x \left[\frac{1}{x} + \log x \right]$$

Example 28.8 Find the derivative of each of the following functions :

(i) $\log \tan x$ (ii) $\log [\cos (\log x)]$

Solution : (i) Let $y = \log \tan x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x)$$

$$= \frac{1}{\tan x} \cdot \sec^2 x$$

$$= \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}$$

$$= \text{cosec } x . \sec x$$

(ii) Let $y = \log [\cos (\log x)]$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos(\log x)} \cdot \frac{d}{dx}[\cos(\log x)]$$

$$= \frac{1}{\cos(\log x)} \cdot \left[-\sin \log x \frac{d}{dx}(\log x) \right]$$

$$= \frac{-\sin(\log x)}{\cos(\log x)} \cdot \frac{1}{x}$$

$$= -\frac{1}{x} \tan(\log x)$$

Example 28.9 Find $\frac{dy}{dx}$, if $y = \log(\sec x + \tan x)$

Solution : $y = \log (\sec x + \text{tax } x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{\sec x + \tan x} \cdot \left[\sec x \tan x + \sec^2 x \right]$$

$$= \frac{1}{\sec x + \tan x} \cdot \sec x \left[\sec x + \tan x\right]$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$= \sec x$$

Example 28.10 Find $\frac{dy}{dx}$, if

$$y = \frac{(4x^2 - 1)(1 + x^2)^{\frac{1}{2}}}{x^3 (x-7)^{\frac{3}{4}}}$$

Solution : Although, you can find the derivative directly using quotient rule (and product rule) but if you take logarithm on both sides, the product changes to addition and division changes to subtraction. This simplifies the process:

$$y = \frac{(4x^2 - 1)(1 + x^2)^{\frac{1}{2}}}{x^3 (x-7)^{\frac{3}{4}}}$$

Taking logarithm on both sides, we get

$$\therefore \qquad \log y = \log \left[\frac{\left(4x^2 - 1\right)\left(1 + x^2\right)^{\frac{1}{2}}}{x^3 \left(x-7\right)^{\frac{3}{4}}} \right]$$

$$\text{or} \qquad \log y = \log\left(4x^2 - 1\right) + \frac{1}{2}\log\left(1 + x^2\right) - 3\log x - \frac{3}{4}\log(x-7)$$

[Using log properties]

Now, taking derivative on both sides, we get

$$\frac{d}{dx}(\log y) = \frac{1}{4x^2 - 1} \cdot 8x + \frac{1}{2(1+x^2)} \cdot 2x - \frac{3}{x} - \frac{3}{4} \cdot \left(\frac{1}{x-7}\right)$$

$$\Rightarrow \qquad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{8x}{4x^2 - 1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4\left(x-7\right)}$$

$$\therefore \qquad \frac{dy}{dx} = y\left[\frac{8x}{4x^2 - 1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4(x-7)}\right]$$

$$= \frac{(4x^2 - 1)\sqrt{1+x^2}}{x^3 (x-7)^{\frac{3}{4}}} \left[\frac{8x}{4x^2 - 1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4(x-7)}\right]$$

CHECK YOUR PROGRESS 28.2

1. Find the derivative of each the functions given below:

   (a) $f(x) = 5\sin x \;\; 2\log x$ (b) $f(x) = \log \cos x$

2. Find $\dfrac{dy}{dx}$, if

   (a) $y = e^{x^2}\log x$ (b) $y = \dfrac{e^{x^2}}{\log x}$

3. Find the derivative of each of the following functions :

   (a) $y = \log(\sin \log x)$ (b) $y = \log \tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$

   (c) $y = \log\left[\dfrac{a + b\tan x}{a - b\tan x}\right]$ (d) $y = \log(\log x)$

4. Find $\dfrac{dy}{dx}$, if

   (a) $y = (1+x)^{\frac{1}{2}}(2-x)^{\frac{2}{3}}(x^2+5)^{\frac{1}{7}}(x+9)^{-\frac{3}{2}}$ (b) $y = \dfrac{\sqrt{x}(1-2x)^{\frac{3}{2}}}{(3+4x)^{\frac{5}{4}}(3-7x^2)^{\frac{1}{4}}}$

28.3 DERIVATIVE OF LOGARITHMIC FUNCTION (CONTINUED)

We know that derivative of the function x^n w.r.t. x is $n x^{n-1}$, where n is a constant. This rule is not applicable, when exponent is a variable. In such cases we take logarithm of the function and then find its derivative.

Therefore, this process is useful, when the given function is of the type $[f(x)]^{g(x)}$. For example,

a^x, x^x etc.

Note : Here f(x) may be constant.

Derivative of a^x w.r.t. x

Let $y = a^x$, $a > 0$

Taking log on both sides, we get

$$\log y = \log a^x = x\log a \qquad [\ \log m^n = n\log m]$$

$$\therefore \quad \frac{d}{dx}(\log y) = \frac{d}{dx}(x\log a) \quad \text{or} \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log a \times \frac{d}{dx}(x)$$

or $$\frac{dy}{dx} = y\log a$$

$$= a^x \log a$$

Thus, $\quad \mathbf{\dfrac{d}{dx}a^x = a^x \log a}, \qquad a > 0$

Example 28.11 Find the derivative of each of the following functions :

(i) $y = x^x$ (ii) $y = x^{\sin x}$

Solution : (i) $y = x^x$

Taking logrithms on both sides, we get

$$\log y = x \log x$$

Taking derivative on both sides, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log x \frac{d}{dx}(x) + x\frac{d}{dx}(\log x) \qquad \text{[Using product rule]}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\log x + x\cdot\frac{1}{x}$$

$$= \log x + 1$$

$$\therefore \qquad \frac{dy}{dx} = y[\log x + 1]$$

Thus, $\quad \dfrac{dy}{dx} = x^x(\log x + 1)$

(ii) $\quad y = x^{\sin x}$

Taking logarithm on both sides, we get

$$\log y = \sin x \log x$$

$$\therefore \qquad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(\sin x \log x)$$

or $\quad \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \cos x.\log x + \sin x\cdot\dfrac{1}{x}$

or $\quad \dfrac{dy}{dx} = y\left[\cos x \log x + \dfrac{\sin x}{x}\right]$

Thus, $\quad \dfrac{dy}{dx} = x^{\sin x}\left[\cos x \log x + \dfrac{\sin x}{x}\right]$

Example 28.12 Find the derivative, if

$$y = (\log x)^x + \left(\sin^{-1} x\right)^{\sin x}$$

Solution : Here taking logarithm on both sides will not help us as we cannot put

$(\log x)^x + (\sin^{-1} x)^{\sin x}$ in simpler form. So we put

$$u = (\log x)^x \qquad \text{and} \qquad v = (\sin^{-1} x)^{\sin x}$$

Then,
$$y = u + v$$

$\therefore$
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{......(i)}$$

Now
$$u = (\log x)^x$$

Taking log on both sides, we have

$$\log u = \log(\log x)^x$$

$\therefore$
$$\log u = x \log(\log x) \qquad \left[\because \log m^n = n \log m\right]$$

Now, finding the derivative on both sides, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = 1 \cdot \log(\log x) + x \frac{1}{\log x} \cdot \frac{1}{x}$$

Thus,
$$\frac{du}{dx} = u\left[\log(\log x) + \frac{1}{\log x}\right]$$

Thus,
$$\frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] \qquad \text{.....(ii)}$$

Also,
$$v = (\sin^{-1} x)^{\sin x}$$

$\therefore$
$$\log v = \sin x \log(\sin^{-1} x)$$

Taking derivative on both sides, we have

$$\frac{d}{dx}(\log v) = \frac{d}{dx}[\sin x \log(\sin^{-1} x)]$$

$$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} + \cos x \cdot \log\left(\sin^{-1} x\right)$$

or,
$$\frac{dv}{dx} = v\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \cdot \log \sin^{-1} x\right]$$

$$= \left(\sin^{-1} x\right)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \log\left(\sin^{-1} x\right)\right] \qquad \text{....(iii)}$$

From (i), (ii) and (iii), we have

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] + \left(\sin^{-1} x\right)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \log \sin^{-1} x\right]$$

Example 28.13 If $x^y = e^{x-y}$, prove that

$$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$$

Solution : It is given that $\qquad x^y = e^{x-y} \qquad$ ...(i)

Taking logarithm on both sides, we get

$$y\log x=(x-y)\log e$$

$$=(x-y)$$

or $\qquad y(1+\log x)=x \qquad [\because \quad \log e=1]$

or $\qquad y=\dfrac{x}{1+\log x} \qquad$ (ii)

Taking derivative with respect to x on both sides of (ii), we get

$$\frac{dy}{dx}=\frac{(1+\log x)\cdot 1-x\left(\frac{1}{x}\right)}{(1+\log x)^2}$$

$$=\frac{1+\log x-1}{(1+\log x)^2}=\frac{\log x}{(1+\log x)^2}$$

Example 28.14 Find , $\dfrac{dy}{dx}$ if

$$e^x\log y=\sin^{-1}x+\sin^{-1}y$$

Solution : We are given that

$$e^x\log y=\sin^{-1}x+\sin^{-1}y$$

Taking derivative with respect to x of both sides, we get

$$e^x\left(\frac{1}{y}\frac{dy}{dx}\right)+e^x\log y=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

or $$\left[\frac{e^x}{y}-\frac{1}{\sqrt{1-y^2}}\right]\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-e^x\log y$$

or $$\frac{dy}{dx}=\frac{y\sqrt{1-y^2}\left[1-e^x\sqrt{1-x^2}\log y\right]}{\left[e^x\sqrt{1-y^2}-y\right]\sqrt{1-x^2}}$$

Example 28.15 Find $\dfrac{dy}{dx}$, if $y=(\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}}$

Solution : We are given that

$$y = (\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}} = (\cos x)^y$$

Taking logarithm on both sides, we get

$$\log y = y \log \cos x$$

Differentiating (i) w.r.t.x, we get

$$\frac{1}{y}\frac{dy}{dx} = y \cdot \frac{1}{\cos x}(-\sin x) + \log(\cos x) \cdot \frac{dy}{dx}$$

or $$\left[\frac{1}{y} - \log(\cos x)\right]\frac{dy}{dx} = -y \tan x$$

or $$\left[1 - y\log(\cos x)\right]\frac{dy}{dx} = -y^2 \tan x$$

or $$\frac{dy}{dx} = \frac{-y^2 \tan x}{1 - y\log(\cos x)}$$

CHECK YOUR PROGRESS 28.3

1. Find the derivative with respect to x of each the following functions :

   (a) $y = 5^x$ (b) $y = 3^x + 4^x$ (c) $y = \sin(5^x)$

2. Find $\frac{dy}{dx}$, if

   (a) $y = x^{2x}$ (b) $y = (\cos x)^{\log x}$ (c) $y = (\log x)^{\sin x}$

   (d) $y = (\tan x)^x$ (e) $y = (1+x^2)^{x^2}$ (f) $y = x^{(x^2 + \sin x)}$

3. Find the derivative of each of the functions given below :

   (a) $y = (\tan x)^{\cot x} + (\cot x)^x$ (b) $y = x^{\log x} + (\sin x)^{\sin^{-1} x}$

   (c) $y = x^{\tan x} + (\sin x)^{\cos x}$ (d) $y = (x)^{x^2} + (\log x)^{\log x}$

4. If $y = (\sin x)^{(\sin x)^{(\sin x)\ldots\ldots\infty}}$, show that

   $$\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y\log(\sin x)}$$

5. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\ldots\infty}}}$, show that

   $$\frac{dy}{dx} = \frac{1}{x(2x-1)}$$

28.4 SECOND ORDER DERIVATIVES

In the previous lesson we found the derivatives of second order of trigonometric and inverse trigonometric functions by using the formulae for the derivatives of trigonometric and inverse trigonometric functions, various laws of derivatives, including chain rule, and power rule discussed earlier in lesson 21. In a similar manner, we will discuss second order derivative of exponential and logarithmic functions :

Example 28.16 Find the second order derivative of each of the following :

(i) e^x (ii) $\cos(\log x)$ (iii) x^x

Solution : (i) Let $y = e^x$

Taking derivative w.r.t. x on both sides, we get $\dfrac{dy}{dx} = e^x$

Taking derivative w.r.t. x on both sides, we get $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(e^x) = e^x$

$$\therefore \qquad \frac{d^2y}{dx^2} = e^x$$

(ii) Let $y = \cos(\log x)$

Taking derivative w.r.t x on both sides, we get

$$\frac{dy}{dx} = -\sin(\log x)\cdot\frac{1}{x} = \frac{-\sin(\log x)}{x}$$

Taking derivative w.r.t. x on both sides, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{\sin(\log x)}{x}\right]$$

or

$$= -\frac{x\cdot\cos(\log x)\cdot\frac{1}{x} - \sin(\log x)}{x^2}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{\sin(\log x) - \cos(\log x)}{x^2}$$

(iii) Let $y = x^x$

Taking logarithm on both sides, we get

$$\log y = x\log x \qquad \text{....(i)}$$

Taking derivative w.r.t. x of both sides, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x = 1 + \log x$$

or

$$\frac{dy}{dx} = y(1+\log x) \qquad \text{....(ii)}$$

Taking derivative w.r.t. x on both sides we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}[y(1+\log x)]$$

$$=y\cdot\frac{1}{x}+(1+\log x)\frac{dy}{dx} \qquad \text{...(iii)}$$

$$=\frac{y}{x}+(1+\log x)y(1+\log x)$$

$$=\frac{y}{x}+(1+\log x)^2y \qquad \text{(Using (ii))}$$

$$=y\left[\frac{1}{x}+(1+\log x)^2\right]$$

$$\therefore \qquad \frac{d^2y}{dx^2}=x^x\left[\frac{1}{x}+(1+\log x)^2\right]$$

Example 28.17 If $y=e^{a\cos^{-1}x}$, show that

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0\cdot$$

Solution : We have, $y=e^{a\cos^{-1}x}$ .....(i)

$$\therefore \qquad \frac{dy}{dx}=e^{a\cos^{-1}x}\cdot\frac{-a}{\sqrt{1-x^2}}$$

$$=-\frac{ay}{\sqrt{1-x^2}} \qquad \text{Using (i)}$$

or $$\left(\frac{dy}{dx}\right)^2=\frac{a^2y^2}{1-x^2}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)^2\left(1-x^2\right)-a^2y^2=0 \qquad \text{.....(ii)}$$

Taking derivative of both sides of (ii), we get

$$\left(\frac{dy}{dx}\right)^2(-2x)+2\left(1-x^2\right)\times\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-a^2\cdot 2y\cdot\frac{dy}{dx}=0$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0 \qquad \left[\text{Dividing through out by } 2\cdot\frac{dy}{dx}\right]$$

CHECK YOUR PROGRESS 28.4

1. Find the second order derivative of each of the following :

   (a) $x^4 e^{5x}$ (b) $\tan\left(e^{5x}\right)$ (c) $\dfrac{\log x}{x}$

2. If $y = a\cos(\log x) + b\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

3. If $y = e^{\tan^{-1}x}$, prove that

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

28.5 DERIVATIVE OF PARAMETRIC FUNCTIONS

Sometimes x and y are two variables such that both are explicitly expressed in terms of a third variable, say t, i.e. if $x = f(t)$ and $y = g(t)$, then such functions are called parametric functions and the third variable is called the parameter.

In order to find the derivative of a function in parametric form, we use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$$

or

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ provided } \frac{dx}{dt} \neq 0$$

Example 28.18 Find $\dfrac{dy}{dx}$, when $x = a\sin t,\ y = a\cos t$

Differentiating w.r. to 't', we get

$$\frac{dx}{dt} = a\cos t \text{ and } \frac{dy}{dt} = -a\sin t$$

Hence, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a\sin t}{a\cos t} = -\tan t$$

Example 28.19 Find $\dfrac{dy}{dx}$, if $x = 2at^2$ and $y = 2at$.

Solution : Given $x = 2at^2$ and $y = 2at$.

Differentiating w.r. to 't', we get

$$\frac{dx}{dt} = 4at \text{ and } \frac{dy}{dt} = 2a$$

Hence $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{4at} = \frac{1}{2t}$$

Example 28.20 Find $\frac{dy}{dx}$, If $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$

Solution : Given
$$x = a(\theta - \sin\theta) \text{ and}$$
$$y = a(1+\cos\theta)$$

Differentiating both w.r. to 'θ', we get

$$\frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = a(-\sin\theta)$$

Hence $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)} = -\cot\theta/2$$

Example 28.21 Find $\frac{dy}{dx}$, if $x = a\cos^3 t$ and $y = a\sin^3 t$

Solution : Given $x = a\cos^3 t$ and $y = a\sin^3 t$

Differentiating both w.r. to 't', we get

$$\frac{dx}{dt} = 3a\cos^2 t\frac{d}{dt}(\cos t) = -3a\cos^2 t\sin t$$

and $$\frac{dy}{dt} = 3a\sin^2 t\frac{d}{dt}(\sin t) = 3a\sin^2 t\cos t$$

Hence $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

Example 28.22 Find $\frac{dy}{dx}$, If $x = a\frac{1-t^2}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$.

Solution : Given $x = a\frac{1-t^2}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$

Differentiating both w.r. to 't', we get

$$\frac{dx}{dt} = a\left\{\frac{(1+t^2).(0-2t)-(1-t^2)(0+2t)}{(1+t^2)^2}\right\} = \frac{-4at}{(1+t^2)^2}$$

and $$\frac{dy}{dt} = 2b\left\{\frac{(1+t^2).(1)-t.(0+2t)}{(1+t^2)^2}\right\} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

Hence $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at} = \frac{-b(1-t^2)}{2at}$$

CHECK YOUR PROGRESS 28.5

Find $\frac{dy}{dx}$, when :

1. $x = 2at^3$ and $y = at^4$
2. $x = a \cos \theta$ and $y = a \sin \theta$
3. $x = 4t$ and $y = \frac{4}{t}$
4. $x = b\sin^2\theta$ and $y = a\cos^2\theta$
5. $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$
6. $x = a\sec\theta$ and $y = b\tan\theta$
7. $x = \frac{3at}{1+t^2}$ and $y = \frac{3at^2}{1+t^2}$
8. $x = \sin 2t$ and $y = \cos 2t$

28.6 SECOND ORDER DERIVATIVE OF PARAMETRIC FUNCTIONS

If two parametric functions $x = f(t)$ and $y = g(t)$ are given then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = h(t) \quad \left(\text{let here } \frac{dx}{dt} \neq 0\right)$$

Hence $$\frac{d^2y}{dx^2} = \frac{d}{dt}\big((h(t)\big) \times \frac{dt}{dx}$$

Example 28.23 Find $\frac{d^2y}{dx^2}$, if $x = at^2$ and $y = 2at$

Solution : Differentiating both w.r. to ‘t’, we get

$$\frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

Differentiating both sides w.r. to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \times \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{t^2} \times \frac{1}{2at} = -\frac{1}{2at^3}$$

Example 28.24 Find $\frac{d^2y}{dx^2}$, if $x = a\sin^3\theta$ and $y = b\cos^3\theta$

Solution : Given $x = a\sin^3\theta$ and $y = b\cos^3\theta$

Differentiating both w.r. to 'θ', we get

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta \text{ and } \frac{dy}{d\theta} = 3b\cos^2\theta(-\sin\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3b\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\frac{b}{a}\cot\theta$$

Differentiating both sides w.r. to 'x', we get

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\frac{d}{dx}(\cot\theta) = \frac{-b}{a}\frac{d}{d\theta}(\cot\theta) \times \frac{d\theta}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a}(-\operatorname{cosec}^2\theta) \times \frac{1}{3a\sin^2\theta\cos\theta}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{b}{3a^2}\operatorname{cosec}^4\theta\sec\theta$$

Example 28.25 If $x = a\sin t$ and $y = b\cos t$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$

Solution : Given $x = a\sin t$ and $y = b\cos t$

Differentiating both w.r. to 't', we get

$$\frac{dx}{dt} = a\cos t \text{ and } \frac{dy}{dt} = -b\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-b\sin t}{a\cos t} = \frac{-b}{a}\tan t$$

Differentiating both sides w.r. to 'x', we get

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\frac{d}{dt}(\tan t)\times\frac{dt}{dx} = \frac{-b}{a}\sec^2 t\times\frac{1}{a\cos t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a^2}\sec^3 t$$

$$\left(\frac{d^2y}{dx^2}\right) at\ t=\frac{\pi}{4} = \frac{-b}{a^2}\sec^3\frac{\pi}{4} = \frac{-b}{a^2}(\sqrt{2})^3 = \frac{-2\sqrt{2}b}{a^2}$$

CHECK YOUR PROGRESS 28.6

Find $\frac{d^2y}{dx^2}$, when

1. $x = 2at$ and $y = at^2$
2. $x = a(t+\sin t)$ and $y = a(1-\cos t)$
3. $x = 10(\theta - \sin\theta)$ and $y = 12(1-\cos\theta)$
4. $x = a\sin t$ and $y = b\cos 2t$
5. $x = a - \cos 2t$ and $y = b - \sin 2t$

LET US SUM UP

- (i) $\frac{d}{dx}(e^x) = e^x$ (ii) $\frac{d}{dx}(a^x) = a^x \log a \quad ; \ a > 0$
- If μ is a derivable function of x, then

  (i) $\log a \cdot \frac{dx}{dx} \ ; a > 0$

  (iii) $\frac{d}{dx}(e^{ax+b}) = e^{ax+b}\cdot a = ae^{ax+b}$
- (i) $\frac{d}{dx}(\log x) = \frac{1}{x}$ (ii) $\frac{d}{dx}(\log x) = \frac{1}{x}\cdot\frac{d\mu}{dx}$, if μ is a derivable function of x.
- (iii) $\frac{d}{dx}\log(ax+b) = \frac{1}{ax+b}\cdot a = \frac{a}{ax+b}$
- If $x = f(t)$ and $y = g(t)$,

  then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$,. provided $\frac{dx}{dt} \neq 0$

- If $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = h(t)$,

  then $\frac{d^2y}{dx^2} = \frac{d}{dt}[h(t)] \times \frac{dt}{dx}$

SUPPORTIVE WEB SITES

http://www.themathpage.com/acalc/exponential.htm

http://www.math.brown.edu/utra/explog.html

http://www.freemathhelp.com/derivative-log-exponent.html

TERMINAL EXERCISE

1. Find the derivative of each of the following functions :

   (a) $(x^x)^x$ (b) $x^{\left(x^x\right)}$

2. Find $\frac{dy}{dx}$, if

   (a) $y = a^{x \log \sin x}$ (b) $y = (\sin x)^{\cos^{-1} x}$

   (c) $y = \left(1 + \frac{1}{x}\right)^{x^2}$ (d) $y = \log\left[e^x \left(\frac{x-4}{x+4}\right)^{\frac{3}{4}}\right]$

3. Find the derivative of each of the functions given below :

   (a) $f(x) = \cos x \log(x) e^{x^2} x^x$ (b) $f(x) = (\sin^{-1} x)^2 \cdot x^{\sin x} \cdot e^{2x}$

4. Find the derivative of each of the following functions :

   (a) $y = (\tan x)^{\log x} + (\cos x)^{\sin x}$ (b) $y = x^{\tan x} + (\sin x)^{\cos x}$

5. Find $\frac{dy}{dx}$, if (a) If $y = \frac{x^4\sqrt{x+6}}{(3x+5)^2}$ (b) If $y = \frac{e^x + e^{-x}}{(e^x - e^{-x})}$

6. Find $\frac{dy}{dx}$, if (a) If $y = a^x \cdot x^a$ (b) $y = 7^{x^2+2x}$

7. Find the derivative of each of the following functions :

(a) $y = x^2 e^{2x} \cos 3x$ (b) $y = \dfrac{2^x \cot x}{\sqrt{x}}$

8. If $y = x^{x^{x^{\cdots\cdots\infty}}}$, prove that $x\dfrac{dy}{dx} = \dfrac{y^2}{1 - y\log x}$

Find derivative of each of the following function

9. $(\sin x)^{\cos x}$

10. $(\log x)^{\log x}$

11. $\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

12. $\left(x + \dfrac{1}{x}\right)^x + x^{x+\frac{1}{x}}$

13. $x = a\left(\cos t + \log\dfrac{t}{2}\right)$ and $y = a\sin t$

14. $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

15. $x = e^t(\sin t + \cos t)$ and $y = e^t(\sin t - \cos t)$

16. $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

17. $x = a\left(t + \dfrac{1}{t}\right)$ and $y = a\left(t - \dfrac{1}{t}\right)$

18. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$

19. If $x = \dfrac{2bt}{1+t^2}$ and $y = \dfrac{a(1-t^2)}{1+t^2}$, find $\dfrac{dy}{dx}$ at $t = 2$.

20. If $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$, prove that $\dfrac{dy}{dx} = -\cot 3t$

21. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, prove that $\dfrac{dy}{dx} = \tan\left(\dfrac{3\theta}{2}\right)$

22. If $x = \cos t$ and $y = \sin t$, prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{3}}$ at $t = \dfrac{2\pi}{3}$

23. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

24. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{2}$

25. If $x = a\sin pt$ and $y = b\cos pt$, find the value of $\dfrac{d^2y}{dx^2}$ at $t = 0$

26. If $x = \log t$ and $y = \dfrac{1}{t}$, find $\dfrac{d^2y}{dx^2}$

27. If $x = a(1 + \cos t)$ and $y = a(t + \sin t)$, find $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{2}$

28. If $x = at^2$ and $y = 2at$, find $\dfrac{d^2y}{dx^2}$.

ANSWERS

CHECK YOUR PROGRESS 28.1

1. (a) $5e^{5x}$ (b) $7e^{7x+4}$ (c) $\sqrt{2}e^{\sqrt{2}x}$ (d) $-\frac{7}{2}e^{-\frac{7}{2}x}$ (e) $2(x+1)e^{x^2+2x}$

2. (a) $\frac{1}{3}e^x$ (b) $\sec^2 x + 2\cos x - 3\sin x - \frac{1}{2}e^x$ (c) $5\cos x - 2e^x$ (d) $e^x - e^{-x}$

3. (a) $\frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$ (b) $e^{\sqrt{\cot x}}\left[\frac{-\cos ec^2 x}{2\sqrt{\cot x}}\right]$

   (c) $e^{x\sin^2 x}[\sin x + 2x\cos x]\sin x$ (d) $e^{x\sec^2 x}\left[\sec^2 x + 2x\sec^2 x\tan x\right]$

4. (a) xe^x (b) $2e^{2x}\sin x(\sin x + \cos x)$

5. (a) $\frac{2x^2 - x + 2}{(x^2+1)^{3/2}}e^{2x}$ (b) $\frac{e^{2x}[(2x-1)\cot x - x\cos ec^2 x]}{x^2}$

CHECK YOUR PROGRESS 28.2

1. (a) $5\cos x - \frac{2}{x}$ (b) $-\tan x$

2. (a) $e^{x^2}\left[2x\log x + \frac{1}{x}\right]$ (b) $\frac{2x^2\log x - 1}{x(\log x)^2}.e^{x^2}$

3. (a) $\frac{\cot(\log x)}{x}$ (b) $\sec x$ (c) $\frac{2ab\sec^2 x}{a^2 - b^2\tan^2 x}$ (d) $\frac{1}{x\log x}$

4. (a) $(1+x)^{\frac{1}{2}}(2-x)^{\frac{2}{3}}(x^2+5)^{\frac{1}{7}}(x+9)^{-\frac{3}{2}} \times \left[\frac{1}{2(1+x)} - \frac{2}{3(2-x)} + \frac{2x}{7(x^2-5)} - \frac{3}{2(x+9)}\right]$

   (b) $\frac{\sqrt{x}(1-2x)^{\frac{3}{2}}}{(3+4x)^{\frac{5}{4}}(3-7x^2)^{\frac{1}{4}}}\left[\frac{1}{2x} - \frac{3}{1-2x} - \frac{5}{3+4x} + \frac{7x}{2(3-7x^2)}\right]$

CHECK YOUR PROGRESS 28.3

1. (a) $5^x\log 5$ (b) $3^x\log 3 + 4^x\log 4$ (c) $\cos 5^x 5^x \log 5$

2. (a) $2x^{2x}(1+\log x)$ (b) $(\cos x)^{\log x}\left[\frac{\log\cos x}{x} - \tan x\log x\right]$

   (c) $(\log x)^{\sin x}\left[\cos x\log(\log x) + \frac{\sin x}{x\log x}\right]$

(d) $(\tan x)^x \left[\log \tan x + \dfrac{x}{\sin x \cos x}\right]$ (e) $(1+x)^{x^2}\left[2x\log(1+x^2)+2\dfrac{x^3}{1+x^2}\right]$

(f) $x^{(x^2+\sin x)}\left[\dfrac{x^2+\sin x}{x}+(2x+\cos x)\log x\right]$

3. (a) $\operatorname{cosec}^2 x(1-\log \tan x)(\tan x)^{\cot x}+\left(\log \cot x - x\operatorname{cosec}^2 x \tan x\right)(\cot x)^x$

(b) $2x^{(\log x-1)}\log x+(\sin x)^{\sin^{-1}x}\left[\cot x \sin^{-1}x+\dfrac{\log \sin x}{\sqrt{1-x^2}}\right]$

(c) $x^{\tan x}\left(\dfrac{\tan x}{x}+\sec^2 x \log x\right)+(\sin x)^{\cos x}\left[\cos x \cot x - \sin x \log \sin x\right]$

(d) $(x)^{x^2}\cdot x(1+2\log x)+(\log x)^{\log x}\left[\dfrac{1+\log(\log x)}{x}\right]$

CHECK YOUR PROGRESS 28.4

1. (a) $e^{5x}\left(25x^4+40x^3+12x^2\right)$ (b) $25e^{5x}\sec^2\left(e^{5x}\right)\left\{1+2e^{5x}\tan e^{5x}\right\}$

(c) $\dfrac{2\log x-3}{x^3}$

CHECK YOUR PROGRESS 28.5

1. $\dfrac{2t}{3}$ 2. $-\cot\theta$ 3. $-\dfrac{1}{t^2}$ 4. $-\dfrac{a}{b}$ 5. $\dfrac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$

6. $\dfrac{b}{a}\operatorname{cosec}\theta$ 7. $\dfrac{2t}{1-t^2}$ 8. $-\tan 2t$

CHECK YOUR PROGRESS 28.6

1. $\dfrac{1}{2a}$ 2. $\dfrac{\sec^4 t/2}{4a}$ 3. $\dfrac{-3}{100}\operatorname{cosec}^4 \theta/2$ 4. $\dfrac{-4b}{a^2}$ 5. $\operatorname{cosec}^3 2t$

TERMINAL EXERCISE

1. (a) $(x^x)^x[x+2x\log x]$ (b) $x^{(x)^x}[x^{x-1}+\log x(\log x+1)x^x]$

2. (a) $a^{x\log\sin x}[\log \sin x + x\cot x]\log a$

(b) $(\sin x)^{\cos^{-1}x}\left[\cos^{-1}x \cot x-\dfrac{\log \sin x}{\sqrt{1-x^2}}\right]$

(c) $\left(1+\frac{1}{x}\right)^{x^2}\left[2x\log\left(x+\frac{1}{x}\right)-\frac{1}{1+\frac{1}{x}}\right]$ (d) $1+\frac{3}{4(x-4)}-\frac{3}{4(x+4)}$

3. (a) $\cos x\log(x)e^{x^2}\cdot x^x\left[-\tan x+\frac{1}{x\log x}+2x+1+\log x\right]$

(b) $(\sin^{-1}x)^2.x^{\sin x}e^{2x}\left[\frac{2}{\sqrt{1-x^2}\sin^{-1}x}+\cos x\log x+\frac{\sin x}{x}+2\right]$

4. (a) $(\tan x)^{\log x}\left[2\operatorname{cosec} 2x\log x+\frac{1}{x}\log\tan x\right]$

$+(\cos x)^{\sin x}[-\sin x\tan x+\cos x\log(\cos x)]$

(b) $x^{\tan x}\left[\frac{\tan x}{x}+\sec^2 x\log x\right]+(\sin x)^{\cos x}[\cot x\cos x-\sin x\log\sin x)]$

5. (a) $\frac{x^4\sqrt{x+6}}{(3x+5)^2}\left[\frac{4}{x}+\frac{1}{2(x+6)}-\frac{6}{(3x+5)}\right]$ (b) $\frac{-4e^{2x}}{(e^{2x}-1)^2}$

6. (a) $a^x.x^{a-1}[a+x\log_e a]$ (b) $7^{x^2+2x}(2x+2)\log_e 7$

7. (a) $x^2e^{2x}\cos 3x\left\{\frac{2}{x}+2-3\tan 3x\right\}$ (b) $\frac{2^x\cot x}{\sqrt{x}}\left[\log 2-2\operatorname{cosec}2x-\frac{1}{2x}\right]$

9. $(\sin x)^{\cos x}[-\sin x\log\sin x+\cos x.\cot x]$

10. $(\log x)^{\log x}\left[\frac{\log(\log x)+1}{x}\right]$

11. $\frac{(x-1)(x-2)}{(x-3)(x-4)}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}\right]$

12. $\left(x+\frac{1}{x}\right)^{x}\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]+x^{x+\frac{1}{x}}\left[\frac{x^2-1}{x^2}\log x+\frac{x^2+1}{x^2}\right]$

13. $\tan t$ 14. $\tan\theta$ 15. $\tan t$ 16. $\frac{-y\log x}{x\log y}$ 17. $\frac{x}{y}$ 18. $-\sqrt{3}$

19. $\frac{4a}{3b}$ 20. $\frac{\sec^3\theta}{a\theta}$ 21. $\frac{1}{a}$ 22. $\frac{-b}{a^2}$ 23. $\frac{1}{t}$ 24. $\frac{-1}{a}$

25. $\frac{-1}{2at^3}$ 26. $\frac{-1}{t}$ 27. -2 28. $\frac{1}{t}$

29

APPLICATIONS OF DERIVATIVES

In the previous lesson, we have learnt that the slope of a line is the tangent of the angle which the line makes with the positive direction of x-axis. It is denoted by the letter 'm'. Thus, if θ is the angle which a line makes with the positive direction of x-axis, then m is given by tan θ.

We have also learnt that the slope m of a line, passing through two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

In this lesson, we shall find the equations of tangents and normals to different curves, using derinatives.

OBJECTIVES

After studying this lesson, you will be able to :

- find rate of change of quantities
- find approximate value of functions
- define tangent and normal to a curve (graph of a function) at a point;
- find equations of tangents and normals to a curve under given conditions;
- define monotonic (increasing and decreasing) functions;
- establish that $\frac{dy}{dx} > 0$ in an interval for an increasing function and $\frac{dy}{dx} < 0$ for a decreasing function;
- define the points of maximum and minimum values as well as local maxima and local minima of a function from the graph;
- establish the working rule for finding the maxima and minima of a function using the first and the second derivatives of the function; and
- work out simple problems on maxima and minima.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of coordinate geometry and
- Concept of tangent and normal to a curve
- Concept of diferential coefficient of various functions
- Geometrical meaning of derivative of a function at a point
- Solution of equetions and the inequations.

29.1 RATE OF CHANGE OF QUANTITIES

Let $y = f(x)$ be a function of x and let there be a small change Δx in x, and the corresponding change in y be Δy.

$\therefore$ Average change in y per unit change in $x = \dfrac{\Delta y}{\Delta x}$

As $\Delta x \to 0$, the limiting value of the average rate of change of y with respect to x.

So the rate of change of y per unit change in x

$$= \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Hence, $\dfrac{dy}{dx}$ represents the rate of change of y with respect to x.

Thus,

The value of $\dfrac{dy}{dx}$ at $x = x_0$ i.e. $\left(\dfrac{dy}{dx}\right)_{x=x_0} = f'(x_0)$

$f'(x_0)$ represent the rate of change of y with respect to x at $x = x_0$.

Further, if two variables x and y are varying one with respect to another variable t i.e. if $y = f(t)$ and $x = g(t)$, then by chain rule.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0$$

Hence, the rate of change y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Example 29.1 Find the rate of change of area of a circle with respect to its variable radius r, when $r = 3$ cm.

Solution : Let A be the area of a circle of radius r,

then $A = \pi r^2$

$\therefore$ The rate of change of area A with respect to its radius r

$$\Rightarrow \quad \frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

when $r = 3$ cm, $\dfrac{dA}{dr} = 2\pi \times 3 = 6\pi$

Hence, the area of the circle is changing at the rate of 6π cm^2/cm

Example 29.2 A balloon which always remains spherical, has a variable diameter $\dfrac{3}{2}(2x+3)$. Determine the rate of change of volume with respect to x.

Solution : Radius (say r) of the spherical balloon $= \dfrac{1}{2}$ (diameter)

$$= \frac{1}{2} \times \frac{3}{2}(2x+3) = \frac{3}{4}(2x+3)$$

Let V be the volume of the balloon, then

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{3}{4}(2x+3)\right)^3$$

$$\Rightarrow \qquad V = \frac{9}{16}\pi(2x+3)^3$$

$\therefore$ The rate of change of volume w.r. to 'x'

$$\frac{dV}{dx} = \frac{9}{16}\pi \times 3(2x+3)^2 \times 2 = \frac{27}{8}\pi(2x+3)^2$$

Hence, the volume is changing at the rate of $\frac{27}{8}\pi(2x+3)^2 \text{unit}^3/\text{unit}$

Example 29.3 A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm.

Solution : Let r be the radius of the spherical balloon and V be its volume at any time t, then

$$V = \frac{4}{3}\pi r^3$$

Diff. w.r. to 't' we get

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right).\frac{dr}{dt}$$

$$= \frac{4}{3}\pi.3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

But $$\frac{dV}{dt} = 900 \text{ cm}^3/\text{sec. (given)}$$

So, $$4\pi r^2\frac{dr}{dt} = 900$$

$$\Rightarrow \qquad \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

when $r = 15$ cm,

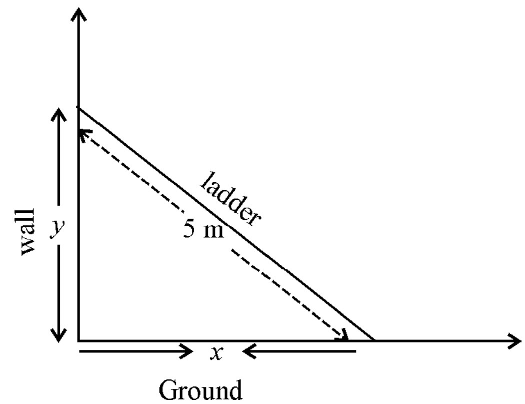
$$\frac{dr}{dt} = \frac{225}{\pi \times 15^2} = \frac{1}{\pi}$$

Hence, the radius of balloon is increasing at the rate of $\frac{1}{\pi}$ cm/sec, when its radius is 15 cm.

Example 29.4 A ladder 5 m long is leaning against a wall. The foot of the ladder is pulled along the ground, away from the wall, at the rate of 2m/sec. How fast is its height on the wall decreasing when the foot of ladder is 4m away from the wall?

Solution : Let the foot of the ladder be at a distance x metres from the wall and y metres be the height of the ladder at any time t, then

$$x^2 + y^2 = 25 \qquad \text{...(i)}$$



Diff. w.r. to 't'. We get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \qquad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

But $\qquad \dfrac{dx}{dt} = 2$ m/sec. (given)

$$\Rightarrow \qquad \frac{dy}{dt} = -\frac{x}{y} \times 2 = -\frac{2x}{y}$$

...(ii)

When $x = 4$m, from (i) $y^2 = 25 - 16 \Rightarrow y = 3$m

Putting $x = 4$m and $y = 3$m in (ii), we get

$$\frac{dy}{dx} = -\frac{2 \times 4}{3} = \frac{-8}{3}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ m/sec.

Example 29.5 The total revenue received from the sale of x units of a product is given by

$$R(x) = 10x^2 + 13x + 24$$

Find the marginal revenue when $x = 5$, where by marginal revenue we mean the rate of change of total revenue w.r. to the number of items sold at an instant.

Solution : Given $R(x) = 10x^2 + 13x + 24$

Since marginal revenue is the rate of change of the revenue with respect to the number of units sold, we have

$$\text{marginal revenue (MR)} = \frac{dR}{dx} = 20x + 13$$

when $x = 5$, $MR = 20 \times 5 + 13 = 113$

Hence, the marginal revenue = ` 113

Example 29.6 The total cost associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

Find the marginal cost when 17 units are produced, where by marginal cost we mean the instantaneous rate of change of the total cost at any level of output.

Solution : Given $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$

Since marginal cost is the rate of change of total cost w.r. to the output, we have

$$\text{Marginal Cost (MC)} = \frac{dC}{dx}$$

$$= 0.007 \times 3x^2 - 0.003 \times 2x + 15$$

$$= 0.021x^2 - 0.006x + 15$$

when x = 17, $\quad MC = 0.021 \times 17^2 - 0.006 \times 17 + 15$

$$= 6.069 - 0.102 + 15$$

$$= 20.967$$

Hence, marginal cost = ` 20.967

CHECK YOUR PROGRESS 29.1

1. The side of a square sheet is increasing at rate of 4 cm per minute. At what rate is the area increasing when the side is 8 cm long?
2. An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 10 cm long.
3. Find the rate of change of the area of a circle with respect to its radius when the radius is 6 cm.
4. The radius of a spherical soap bubble is increasing at the rate of 0.2 cm/sec. Find the rate of increase of its surface area, when the radius is 7 cm.
5. Find the rate of change of the volume of a cube with respect to its edge when the edge is 5 cm.

29.2 APPROXIMATIONS

In this section, we shall give a meaning to the symbols dx and dy in such a way that the original meaning of the symbol $\frac{dy}{dx}$ coincides with the quotient when dy is divided by dx.

Let $y = f(x)$ be a function of x and Δx be a small change in x and let Δy be the corresponding change in y. Then,

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

$$\Rightarrow \qquad \frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \varepsilon, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

$$\Rightarrow \qquad \Delta y = \frac{dy}{dx}\Delta x + \varepsilon \Delta x$$

$\because$ $\varepsilon \Delta x$ is a very-very small quantity that can be neglected, therefore

we have $\qquad \Delta y = \dfrac{dy}{dx}\Delta x$, approximately

This formula is very useful in the calculation of small change (or errors) in dependent variable corresponding to small change (or errors) in the independent variable.

SOME IMPORTANT TERMS

ABSOLUTE ERROR : The error Δx in x is called the absolute error in x.

RELATIVE ERROR : If Δx is an error in x, then $\dfrac{\Delta x}{x}$ is called relative error in x.

PERCENTAGE ERROR : If Δx is an error in x, then $\dfrac{\Delta x}{x} \times 100$ is called percentage error in x.

Note : We have $\Delta y = \dfrac{dy}{dx}\Delta x + \varepsilon.\Delta x$

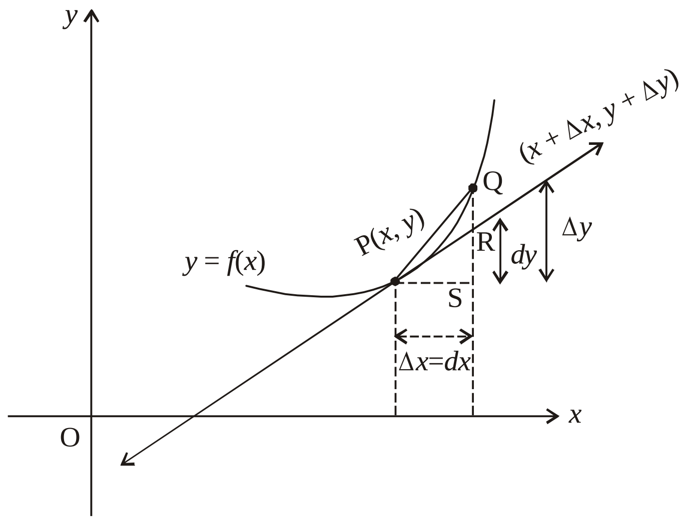
$\because$ $\varepsilon.\Delta x$ is very smal, therefore principal value of $\Delta y = \dfrac{dy}{dx}\Delta x$ which is called differential of y.

i.e. $\qquad \Delta y = \dfrac{dy}{dx}.\Delta x$

So, the differential of x is given by

$$dx = \frac{dx}{dx}.\Delta x = \Delta x$$

Hence, $\qquad dy = \dfrac{dy}{dx}dx$



To understand the geometrical meaning of dx, Δx, dy and Δy. Let us focus our attention to the portion of the graph of $y = f(x)$ in the neighbourhood of the point P(x, y) where a tangent can be drawn the curve. If Q$(x + \Delta x, y + \Delta y)$ be another point $(\Delta x \neq 0)$ on the curve, then the slope of line PQ will be $\frac{\Delta y}{\Delta x}$ which approaches the limiting value $\frac{dy}{dx}$ (slope of tangent at P).

Therefore, when $\Delta x \to 0$, Δy is approximately equal to dy.

Example 29.7 Using differentials, find the approximate value of $\sqrt{25.3}$

Solution : Let $y = \sqrt{x}$

Differentiating w.r. to 'x' we get

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Take $x = 25$ and $x + \Delta x = 25.3$, then $dx = \Delta x = 0.3$ when x = 25, $y = \sqrt{25} = 5$

$$\Delta y = \frac{dy}{dx}\Delta x = \frac{1}{2\sqrt{x}}\Delta x = \frac{1}{2\sqrt{25}} \times 0.3 = \frac{1}{10} \times 0.3 = 0.03$$

$\Rightarrow \Delta y = 0.03$ ($\because$ dy is approximately equal to Δy)

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{25.3}$$

$$\Rightarrow \qquad \sqrt{25.3} = 5 + 0.03 = 5.03 \text{ approximately}$$

Example 29.8 Using differentials find the approximate value of $(127)^{\frac{1}{3}}$

Solution : Take $y = x^{\frac{1}{3}}$

Let $x = 125$ and $x + \Delta x = 127$, then $dx = \Delta x = 2$

When $x = 125$, $y = (125)^{\frac{1}{3}} = 5$

Now $$y = x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3x^{2/3}}$$

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{2/3}}dx = \frac{1}{3(125)^{2/3}} \times 2 = \frac{2}{75}$$

$\Rightarrow$ $$\Delta y = \frac{2}{75}$$

($\because$ $\Delta y = dy$)

Hence, $$(127)^{\frac{1}{3}} = y + \Delta y = 5 + \frac{2}{75} = 5.026 \text{ (Approximate)}$$

Example 29.9 Find the approximate value of $f(3.02)$, where

$$f(x) = 3x^2 + 5x + 3$$

Solution : Let x = 3 and $x + \Delta x = 3.02$, then $dx = \Delta x = 0.02$

We have $$f(x) = 3x^2 + 5x + 3$$

when $x = 3$

$\Rightarrow$ $$f(3) = 3(3)^2 + 5(3) + 3 = 45$$

Now $y = f(x)$

$\Rightarrow$ $$\Delta y = \frac{dy}{dx}\Delta x = (6x + 5)\Delta x$$

$\Rightarrow$ $$\Delta y = (6 \times 3 + 5) \times 0.02 = 0.46$$

$\therefore$ $f(3.02) = f(x + \Delta x) = y + \Delta y = 45 + 0.46 = 45.46$

Hence, the approximate value of $f(3.02)$ is 45.46.

Example 29.10 If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

Solution : Let r be the radius of the sphere and Δr be the error in measuring the radius. Then

$r = 9$ cm and $\Delta r = 0.03$ cm

Let S be the surface area of the sphere. Then

$S = 4\pi r^2$

$\Rightarrow$ $$\frac{dS}{dr} = 4\pi \times 2r = 8\pi r$$

$$\left(\frac{dS}{dr}\right)_{\text{at } r=9} = 8\pi \times (9) = 72\pi$$

Let ΔS be the error in S, then

$$\Delta S = \frac{dS}{dr}\Delta r = 72\pi \times 0.03 = 2.16\pi \text{ cm}^2$$

Hence, approximate error in calculating the surface area is 2.16π cm^2.

Example 29.11 Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%.

Solution : Let Δx be the change in x and ΔV be the corresponding change in V.

Given that $\frac{\Delta x}{x} \times 100 = 2 \Rightarrow \Delta x = \frac{2x}{100}$

we have $V = x^3$

$$\Rightarrow \quad \frac{dV}{dx} = 3x^2$$

Now $\Delta V = \frac{dV}{dx}\Delta x$

$$\Rightarrow \quad \Delta V = 3x^2 \times \frac{2x}{100}$$

$$\Rightarrow \quad \Delta V = \frac{6}{100}.V$$

Hence, the approximate change in volume is 6%.

CHECK YOUR PROGRESS 29.2

1. Using differentials, find the approximate value of $\sqrt{36.6}$.
2. Using differentials, find the appoximate value of $(25)^{\frac{1}{3}}$.
3. Using differentials, find the approximate value of $(15)^{\frac{1}{4}}$.
4. Using differentials, find the approximate value of $\sqrt{26}$.
5. If the radius of a sphere is measured as 7 m with an error of 0.02 m, find the approximate error in calculating its volume.
6. Find the percentage error in calculating the volume of a cubical box if an error of 1% is made in measuring the length of edges of the box.

29.3 SLOPE OF TANGENT AND NORMAL

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be a point on it then the slope PT at $P(x_1, y_1)$ is given by

$$\left(\frac{dy}{dx}\right) \text{at } (x_1, y_1) \qquad ....(i)$$

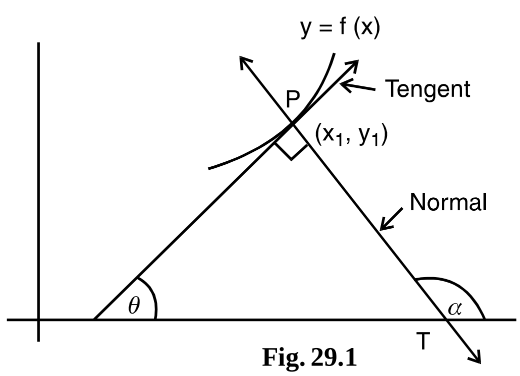


Fig. 29.1

and (i) is equal to $\tan\theta$

We know that a normal to a curve is a line perpendicular to the tangent at the point of contact

We know that $\alpha = \frac{\pi}{2} + \theta$ (From Fig. 10.1)

$$\Rightarrow \qquad \tan\alpha = \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$= -\frac{1}{\tan\theta}$$

$\therefore$ Slope of normal $= -\frac{1}{m} = \frac{-1}{\left(\frac{dy}{dx}\right)}$ at (x_1, y_1) or $-\left(\frac{dx}{dy}\right)$ at (x_1, y_1)

Note

1. The tangent to a curve at any point will be parallel to x-axis if $\theta = 0$, i.e, the derivative at the point will be zero.

   i.e. $\left(\frac{dx}{dy}\right)$ at $(x_1, y_1) = 0$

2. The tangent at a point to the curve $y = f(x)$ will be parallel to y-axis if $\frac{dy}{dx} = 0$ at that point.

Let us consider some examples :

Example 29.12 Find the slope of tangent and normal to the curve

$$x^2 + x^3 + 3xy + y^2 = 5 \text{ at } (1, 1)$$

Solution : The equation of the curve is

$$x^2 + x^3 + 3xy + y^2 = 5 \qquad .....(i)$$

Differentialing (i),w.r.t. x, we get

$$2x + 3x^2 + 3\left[x\frac{dy}{dx} + y.1\right] + 2y\frac{dy}{dx} = 0 \qquad .....(ii)$$

Substituting $x = 1, \quad y = 1$, in (ii), we get

$$2\times1 + 3\times1 + 3\left[\frac{dy}{dx} + 1\right] + 2\frac{dy}{dx} = 0$$

or $\quad 5\frac{dy}{dx} = -8 \Rightarrow \quad \frac{dy}{dx} = -\frac{8}{5}$

$\therefore$ The slope of tangent to the curve at (1, 1) is $-\frac{8}{5}$

$\therefore$ The slope of normal to the curve at (1, 1) is $\frac{5}{8}$

Example 29.13 Show that the tangents to the curve $y=\frac{1}{6}\left[3x^5+2x^3-3x\right]$ at the points $x=\pm 3$ are parallel.

Solution : The equation of the curve is $y=\frac{3x^5+2x^3-3x}{6}$ .....(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=\frac{\left(15x^4+6x^2-3\right)}{6}$$

$$\left(\frac{dy}{dx}\right)\underset{x=3}{\text{at}}=\frac{\left[15(3)^4+6(3)^2-3\right]}{6}$$

$$=\frac{1}{6}\left[15\times 9\times 9+54-3\right]$$

$$=\frac{3}{6}\left[405+17\right]=211$$

$$\left(\frac{dy}{dx}\right)\text{at}\, x=-3=\frac{1}{6}\left[15(-3)^4+6(-3)^2-3\right]=211$$

$\therefore$ The tangents to the curve at $x=\pm 3$ are parallel as the slopes at $x=\pm 3$ are equal.

Example 29.14 The slope of the curve $6y^3=px^2+q$ at $(2,-2)$ is $\frac{1}{6}$.

Find the values of p and q.

Solution : The equation of the curve is

$$6y^3=px^2+q \qquad .....(i)$$

Differentiating (i) w.r.t. x, we get

$$18y^2\frac{dy}{dx}=2px \qquad .....(ii)$$

Putting $x=2,\ y=-2,$ we get

$$18(-2)^2\frac{dy}{dx}=2p\cdot 2=4p$$

$$\therefore \qquad \frac{dy}{dx} = \frac{p}{18}$$

It is given equal to $\frac{1}{6}$

$$\therefore \qquad \frac{1}{6} = \frac{p}{18} \Rightarrow p = 3$$

$\therefore$ The equation of curve becomes

$$6y^3 = 3x^2 + q$$

Also, the point (2, –2) lies on the curve

$$\therefore \qquad 6(-2)^3 = 3(2)^2 + q$$

$$\Rightarrow \qquad -48 - 12 = q \quad \text{or} \quad q = -60$$

$\therefore$ The value of $p = 3,\ q = -60$

CHECK YOUR PROGRESS 29.3

1. Find the slopes of tangents and normals to each of the curves at the given points :

   (i) $y = x^3 - 2x$ at $x = 2$ (ii) $x^2 + 3y + y^2 = 5$ at (1, 1)

   (iii) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$

2. Find the values of p and q if the slope of the tangent to the curve $xy + px + qy = 2$ at (1, 1) is 2.

3. Find the points on the curve $x^2 + y^2 = 18$ at which the tangents are parallel to the line x+y = 3.

4. At what points on the curve $y = x^2 - 4x + 5$ is the tangent perpendiculat to the line $2y + x - 7 = 0$.

29.4 EQUATIONS OF TANGENT AND NORMAL TO A CURVE

We know that the equation of a line passing through a point (x_1, y_1) and with slope m is

$$y - y_1 = m(x - x_1)$$

As discussed in the section before, the slope of tangent to the curve y = f (x) at (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)$ at (x_1, y_1) and that of normal is $\left(-\frac{dx}{dy}\right)$ at (x_1, y_1)

$\therefore$ Equation of tangent to the curve y = f (x) at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} [x - x_1]$$

And, the equation of normal to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{-1}{\frac{dy}{dx}}\right)_{(x_1, y_1)} [x - x_1]$$

Note

(i) The equation of tangent to a curve is parallel to x-axis if $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$. In that case the equation of tangent is $y = y_1$.

(ii) In case $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \to \infty$, the tangent at (x_1, y_1) is parallel to $y-\text{axis}$ and its equation is $x = x_1$

Let us take some examples and illustrate

Example 29.15 Find the equation of the tangent and normal to the circle $x^2 + y^2 = 25$ at the point (4, 3)

Solution : The equation of circle is

$$x^2 + y^2 = 25 \qquad \text{....(i)}$$

Differentialing (1), w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-x}{y}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{(4,3)} \text{at} = -\frac{4}{3}$$

$\therefore$ Equation of tangent to the circle at (4, 3) is

$$y - 3 = -\frac{4}{3}(x - 4)$$

or $\qquad 4(x-4) + 3(y-3) = 0 \qquad$ or, $\qquad 4x + 3y = 25$

Also, slope of the normal $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(4,3)}} = \dfrac{3}{4}$

$\therefore$ Equation of the normal to the circle at (4,3) is

$$y - 3 = \frac{3}{4}(x - 4)$$

or $$4y - 12 = 3x - 12$$

$\Rightarrow$ $$3x = 4y$$

$\therefore$ Equation of the tangent to the circle at (4,3) is $4x+3y = 25$

Equation of the normal to the circle at (4,3) is $3x = 4y$

Example 29.16 Find the equation of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at the point (x_1, y_1) where $y_1 > 0$ and $x_1 = 2$

Solution : The equation of curve is

$$16x^2 + 9y^2 = 144 \qquad .....(i)$$

Differentiating (i), w.r.t. x we get

$$32x + 18y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{16x}{9y}$$

As $x_1 = 2$ and (x_1, y_1) lies on the curve

$\therefore$ $$16(2)^2 + 9(y^2) = 144$$

$\Rightarrow$ $$y^2 = \frac{80}{9} \Rightarrow y = \pm\frac{4}{3}\sqrt{5}$$

As $$y_1 > 0 \Rightarrow y = \frac{4}{3}\sqrt{5}$$

$\therefore$ Equation of the tangent to the curve at $\left(2, \frac{4}{3}\sqrt{5}\right)$ is

$$y - \frac{4}{3}\sqrt{5} = \left(-\frac{16x}{9y}\right)_{\text{at}\left(2, \frac{4\sqrt{5}}{3}\right)}[x - 2]$$

or $$y-\frac{4}{3}\sqrt{5}=-\frac{16}{9}\cdot\frac{2\times3}{4\sqrt{5}}(x-2)$$ or $$y-\frac{4}{3}\sqrt{5}+\frac{8}{3\sqrt{5}}(x-2)=0$$

or $$3\sqrt{5}y-\frac{4}{3}\sqrt{5}\cdot3\sqrt{5}+8(x-2)=0$$

$$3\sqrt{5}y-20+8x-16=0$$ or $$3\sqrt{5}y+8x=36$$

Also, equation of the normal to the curve at $\left(2,\frac{4}{3}\sqrt{5}\right)$ is

$$y-\frac{4}{3}\sqrt{5}=\left(\frac{9y}{16x}\right)_{\text{at}\left(2,\frac{4}{3}\sqrt{5}\right)}[x-2]$$

$$y-\frac{4}{3}\sqrt{5}=\frac{9}{16}\times\frac{2\sqrt{5}}{3}(x-2)$$

$$y-\frac{4}{3}\sqrt{5}=\frac{3\sqrt{5}}{8}(x-2)$$

$$3\times8(y)-32\sqrt{5}=9\sqrt{5}(x-2)$$

$$24y-32\sqrt{5}=9\sqrt{5}\,x-18\sqrt{5}$$ or $$9\sqrt{5}x-24y+14\sqrt{5}=0$$

Example 29.17 Find the points on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which the tangents are parallel to x-axis.

Solution : The equation of the curve is

$$\frac{x^2}{9}-\frac{y^2}{16}=1 \quad \text{....(i)}$$

Differentiating (i) w.r.t. x we get

$$\frac{2x}{9}-\frac{2y}{16}\cdot\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=\frac{16x}{9y}$$

For tangent to be parallel to x-axis, $\frac{dy}{dx}=0$

$$\Rightarrow \quad \frac{16x}{9y}=0 \quad \Rightarrow \quad x=0$$

Putting $x=0$ in (i), we get $y^2=-16 \quad y=\pm4i$

This implies that there are no real points at which the tangent to $\frac{x^2}{9}-\frac{y^2}{16}=1$ is parallel to x-axis.

Example 29.18 Find the equation of all lines having slope – 4 that are tangents to the curve $y=\frac{1}{x-1}$

Solution : $$y=\frac{1}{x-1} \qquad .....(i)$$

$\therefore$ $$\frac{dy}{dx}=-\frac{1}{(x-1)^2}$$

It is given equal to – 4

$\therefore$ $$\frac{-1}{(x-1)^2}=-4$$

$\Rightarrow$ $$(x-1)^2=\frac{1}{4}, \quad \Rightarrow \quad x=1\pm\frac{1}{2}\Rightarrow x=\frac{3}{2},\frac{1}{2}$$

Substituting $x=\frac{1}{2}$ in (i), we get

$$y=\frac{1}{\frac{1}{2}-1}=\frac{1}{-\frac{1}{2}}=-2$$

When $$x=\frac{3}{2}, \quad y=2$$

$\therefore$ The points are $\left(\frac{3}{2},2\right),\left(\frac{1}{2},-2\right)$

$\therefore$ The equations of tangents are

(a) $$y-2=-4\left(x-\frac{3}{2}\right), \quad \Rightarrow \quad y-2=-4x+6 \quad \text{or} \quad 4x+y=8$$

(b) $$y+2=-4\left(x-\frac{1}{2}\right)$$

$\Rightarrow$ $$y+2=-4x+2 \quad \text{or} \quad 4x+y=0$$

Example 29.19 Find the equation of the normal to the curve $y=x^3$ at (2 , 8)

Solution : $$y=x^3 \qquad \Rightarrow \frac{dy}{dx}=3x^2$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{at\,x=2}=12$$

$\therefore$ Slope of the normal $= -\dfrac{1}{12}$

$\therefore$ Equation of the normal is

$$y-8=-\frac{1}{12}(x-2)$$

or $\quad 12(y-8)+(x-2)=0 \quad$ or $\quad x+12y=98$

CHECK YOUR PROGRESS 29.4

1. Find the equation of the tangent and normal at the indicated points :

   (i) $y=x^4-6x^3+13x^2-10x+5$ at (0, 5)

   (ii) $y=x^2$ at $(1, 1)$

   (iii) $y=x^3-3x+2$ at the point whose x–coordinate is 3

2. Find the equation of the targent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at (x_1, y_1)

3. Find the equation of the tangent to the hyperbola

   $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \quad$ at (x_0, y_0)

4. Find the equation of normals to the curve

   $y=x^3+2x+6$ which are parallel to the line $x+14y+4=0$

5. Prove that the curves $x=y^2$ and $xy=k$ cut at right angles if $8k^2=1$

29.5 Mathematical formulation of Rolle's Theorem

Let f be a real function defined in the closed interval [a, b] such that

(i) f is continuous in the closed interval [a, b]

(ii) f is differentiable in the open inteval (a, b)

(iii) f (a) = f (b)

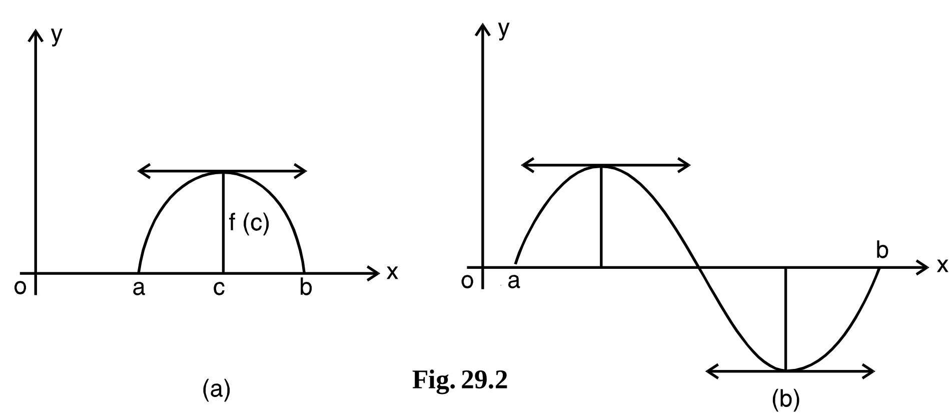


Fig. 29.2

Then there is at least one point c in the open inteval (a, b) such that $f'(c)=0$

Remarks

(i) The remarks "at least one point" says that there can be more than one point $c \in (a, b)$ such that $f'(c)=0$.

(ii) The condition of continuity of f on [a, b] is essential and can not be relaxed

(iii) The condition of differentiability of f on (a,b) is also essential and can not be relaxed.

For example $f(x)=|x|, x \in [-1,1]$ is continuous on [–1,1] and differentiable on (–1, 1) and Rolle's Theorem is valid for this

Let us take some examples

Example 29.20 Verify Rolle's for the function

$$f(x)=x(x-1)(x-2),\ x \in [0, 2]$$

Solution : $$f(x)=x(x-1)(x-2)$$

$$=x^3-3x^2+2x$$

(i) $f(x)$ is a polynomial function and hence continuous in [0, 2]

(ii) $f(x)$ is differentiable on $(0,2)$

(iii) Also $f(0)=0$ and $f(2)=0$

$\therefore$ $$f(0)=f(2)$$

$\therefore$ All the conditions of Rolle's theorem are satisfied.

Also, $$f'(x)=3x^2-6x+2$$

$\therefore f'(c)=0$ gives $3c^2-6c+2=0 \quad \Rightarrow \quad c=\dfrac{6\pm\sqrt{36-24}}{6}$

$\Rightarrow$ $$c=1\pm\frac{1}{\sqrt{3}}$$

We see that both the values of c lie in (0, 2)

Example 29.21 Discuss the applicability of Rolle's Theorem for

$$f(x) = \sin x - \sin 2x\,,\ x \in [0, \pi] \qquad ....(i)$$

(i) is a sine function. It is continuous and differentiable on $(0, \pi)$

Again, we have, $f(0) = 0$ and $f(\pi) = 0$

$\Rightarrow \qquad f(\pi) = f(0) = 0$

$\therefore$ All the conditions of Rolle's theorem are satisfied

Now $\qquad f'(c) = 2\left[2\cos^2 c - 1\right] - \cos c = 0$

or $\qquad 4\cos^2 c - \cos c - 2 = 0$

$$\therefore \qquad \cos c = \frac{1 \pm \sqrt{1+32}}{8}$$

$$= \frac{1 \pm \sqrt{33}}{8}.$$

As $\sqrt{33} < 6$

$$\therefore \qquad \cos c < \frac{7}{8} = 0.875$$

which shows that c lies between 0 and π

CHECK YOUR PROGRESS 29.5

Verify Rolle's Theorem for each of the following functions :

(i) $f(x) = \frac{x^3}{3} - \frac{5x^2}{3} + 2x\,,\ x \in [0, 3]$ (ii) $f(x) = x^2 - 1$ on $[-1, 1]$

(iii) $f(x) = \sin x + \cos x - 1$ on $\left(0, \frac{\pi}{2}\right)$ (iv) $f(x) = \left(x^2 - 1\right)\left(x - 2\right)$ on $[-1, 2]$

29.6 LANGRANGE'S MEAN VALUE THEOREM

This theorem improves the result of Rolle's Theorem saying that it is not necessary that tangent may be parallel to x-axis. This theorem says that the tangent is parallel to the line joining the end points of the curve. In other words, this theorem says that there always exists a point on the graph, where the tangent is parallel to the line joining the end-points of the graph.

29.6.1 Mathematical Formulation of the Theorem

Let f be a real valued function defined on the closed interval [a, b] such that

(a) f is continuous on [a, b], and

(b) f is differentiable in (a, b)

(c) $f(b) \neq f(a)$

then there exists a point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remarks

When $f(b) = f(a)$, $f'(c) = 0$ and the theorem reduces to Rolle's Theorem

Let us consider some examples

Example 29.22 Verify Langrange's Mean value theorem for

$$f(x) = (x-3)(x-6)(x-9) \text{ on } [3, 5]$$

Solution : $f(x) = (x-3)(x-6)(x-9)$

$$= (x-3)(x^2 - 15x + 54)$$

or $$f(x) = x^3 - 18x^2 + 99x - 162 \quad \text{...(i)}$$

(i) is a polynomial function and hence continuous and differentiable in the given interval

Here, $f(3) = 0$, $f(5) = (2)(-1)(-4) = 8$

$\therefore$ $f(3) \neq f(5)$

$\therefore$ All the conditions of Mean value Theorem are satisfied

$\therefore$ $$f'(c) = \frac{f(5) - f(3)}{5 - 3} = \frac{8 - 0}{2} = 4$$

Now $$f'(x) = 3x^2 - 36x + 99$$

$\therefore$ $3c^2 - 36c + 99 = 4$ or $3c^2 - 36c + 95 = 0$

$\therefore$ $$c = \frac{36 \pm \sqrt{1296 - 1140}}{6} = \frac{36 \pm 12.5}{6}$$

$$= 8.08 \text{ or } 3.9$$

$$c = 3.9 \in (3, 5)$$

$\therefore$ Langranges mean value theorem is verified

Example 29.23 Find a point on the parabola $y = (x-4)^2$ where the tangent is parallel to the chord joining $(4, 0)$ and $(5, 1)$

Solution : Slope of the tangent to the given curve at any point is given by $(f'(x))$ at that point.

$$f'(x) = 2(x-4)$$

Slope of the chord joining $(4, 0)$ and $(5, 1)$ is

$$\frac{1-0}{5-4}=1 \qquad \left[\because m=\frac{y_2-y_1}{x_2-x_1}\right]$$

$\therefore$ According to mean value theorem

$$2(x-4)=1 \qquad \text{or} \qquad (x-4)=\frac{1}{2}$$

$$\Rightarrow \qquad x=\frac{9}{2}$$

which lies between 4 and 5

Now $\qquad y=(x-4)^2$

When $\qquad x=\frac{9}{2},\ y=\left(\frac{9}{2}-4\right)^2=\frac{1}{4}$

$\therefore$ The required point is $\left(\frac{9}{2},\frac{1}{4}\right)$

CHECK YOUR PROGRESS 29.6

1. Check the applicability of Mean Value Theorem for each of the following functions :

   (i) $f(x)=3x^2-4$ on $[2,3]$

   (ii) $f(x)=\log x$ on $[1,2]$

   (iii) $f(x)=x+\frac{1}{x}$ on $[1,3]$

   (iv) $f(x)=x^3-2x^2-x+3$ on $[0,1]$

2. Find a point on the parabola $y=(x+3)^2$, where the tangent is parallel to the chord joining $(3,0)$ and $(-4,1)$

29.7 INCREASING AND DECREASING FUNCTIONS

You have already seen the common trends of an increasing or a decreasing function. Here we will try to establish the condition for a function to be an increasing or a decreasing.

Let a function $f(x)$ be defined over the closed interval [a,b].

Let $x_1, x_2 \in [a, b]$, then the function f(x) is said to be an increasing function in the given interval if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$. It is said to be strictly increasing if $f(x_2) > f(x_1)$ for all $x_2 > x_1$, $x_1, x_2 \in [a, b]$.

In Fig. 29.3, sin x increases from -1 to +1 as x increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

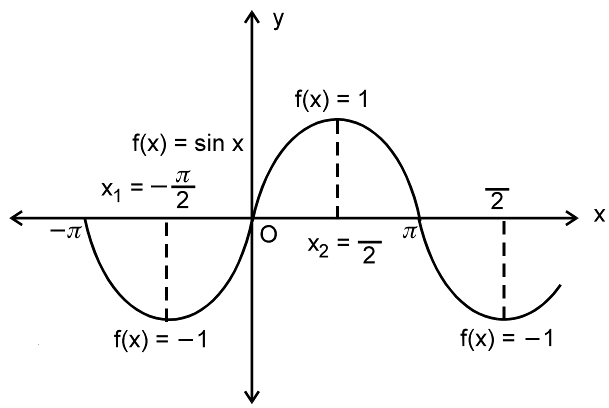


Fig. 29. 3

Note : A function is said to be an increasing function in an interval if $f(x+h) > f(x)$ for all x belonging to the interval when h is positive.

A function f(x) defined over the closed interval [a, b] is said to be a decreasing function in the given interval, if $f(x_2) \leq f(x_1)$, whenever $x_2 > x_1$, $x_1, x_2 \in [a, b]$. It is said to be strictly decreasing if $f(x_1) > f(x_2)$ for all $x_2 > x_1$, $x_1, x_2 \in [a, b]$.

In Fig. 29.4, cos x decreases from 1 to -1 as x increases from 0 to π.

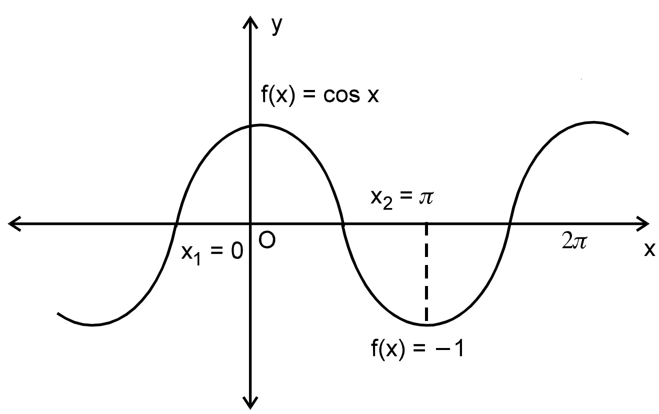


Fig. 29.4

Note : A function is said to be a decreasing in an internal if $f(x+h) < f(x)$ for all x belonging to the interval when h is positive.

29.7.1 MONOTONIC FUNCTIONS

Let x_1, x_2 be any two points such that $x_1 < x_2$ in the interval of definition of a function f(x). Then a function f(x) is said to be monotonic if it is either increasing or decreasing. It is said to be monotonically increasing if $f(x_2) \geq f(x_1)$ for all $x_2 > x_1$ belonging to the interval and monotonically decreasing if $f(x_1) \geq f(x_2)$.

Example 29.24 Prove that the function $f(x) = 4x + 7$ is monotonic for all values of $x \in R$.

Solution : Consider two values of x say $x_1, x_2 \in R$

such that $$x_2 > x_1 \qquad .....(1)$$

Multiplying both sides of (1) by 4, we have $4x_2 > 4x_1$ .....(2)

Adding 7 to both sides of (2), to get

$$4x_2 + 7 > 4x_1 + 7$$

We have $$f(x_2) > f(x_1)$$

Thus, we find $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Hence the given function $f(x) = 4x + 7$ is monotonic function. (monotonically increasing).

Example 29.25 Show that

$$f(x) = x^2$$

is a strictly decreasing function for all $x < 0$.

Solution : Consider any two values of x say x_1, x_2 such that

$$x_2 > x_1, \qquad x_1, x_2 < 0 \qquad ....(i)$$

Order of the inequality reverses when it is multiplied by a negative number. Now multiplying (i) by x_2, we have

$$x_2 \cdot x_2 < x_1 \cdot x_2$$

or, $$x_2^2 < x_1 x_2 \qquad ......(ii)$$

Now multiplying (i) by x_1, we have

$$x_1 \cdot x_2 < x_1 \cdot x_1$$

or, $$x_1 x_2 < x_1^2 \qquad .....(iii)$$

From (ii) and (iii), we have

$$x_2^2 < x_1 x_2 < x_1^2$$

or, $$x_2^2 < x_1^2$$

or, $$f(x_2) < f(x_1) \quad .....(iv)$$

Thus, from (i) and (iv), we have for

$$x_2 > x_1,$$

$$f(x_2) < f(x_1)$$

Hence, the given function is strictly decreasing for all $x < 0$.

CHECK YOUR PROGRESS 29.7

1. (a) Prove that the function

$$f(x) = 3x + 4$$

is monotonic increasing function for all values of $x \in R$.

(b) Show that the function

$$f(x) = 7 - 2x$$

is monotonically decreasing function for all values of $x \in R$.

(c) Prove that $f(x) = ax + b$ where a, b are constants and $a > 0$ is a strictly increasing function for all real values of x.

2. (a) Show that $f(x) = x^2$ is a strictly increasing function for all real $x > 0$.

(b) Prove that the function $f(x) = x^2 - 4$ is monotonically increasing for $x > 2$ and monotonically decreasing for $-2 < x < 2$ where $x \in R$.

Theorem 1 : If f(x) is an increasing function on an open interval]a, b[, then its derivative f '(x) is positive at this point for all $x \in [a, b]$.

Proof : Let (x, y) or [x, f (x)] be a point on the curve y = f (x)

For a positive δx, we have

$$x + \delta x > x$$

Now, function f(x) is an increasing function

$\therefore$ $$f(x + \delta x) > f(x)$$

or, $$f(x + \delta x) - f(x) > 0$$

or, $$\frac{f(x + \delta x) - f(x)}{\delta x} > 0 \quad [\because \delta x > 0]$$

Taking δx as a small positive number and proceeding to limit, when $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} > 0$$

or, $$f'(x) > 0$$

Thus, if $y = f(x)$ is an increasing function at a point, then f '(x) is positive at that point.

Theorem 2 : If f (x) is a decreasing function on an open interval]a, b[then its derivative f '(x) is negative at that point for all $x \in [a, b]$.

Proof : Let (x, y) or [x, f(x)] be a point on the curve y = f(x)

For a positive δx, we have $\quad x + \delta x > x$

Since the function is a decreasing function

$\therefore \qquad f(x + \delta x) < f(x) \qquad \delta x > 0$

or, $\qquad f(x + \delta x) - f(x) < 0$

Dividing by δx, we have $\qquad \dfrac{f(x + \delta x) - f(x)}{\delta x} < 0 \quad \delta x > 0$

or, $\qquad \lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x} < 0$

or, $\qquad f'(x) < 0$

Thus, if $y = f(x)$ is a decreasing function at a point, then, $f'(x)$ is negative at that point.

Note : If $f(x)$ is derivable in the closed interval [a,b], then f (x) is

(i) increasing over [a,b], if f '(x)>0 in the open interval]a,b[

(ii) decreasing over [a,b], if f '(x)<0 in the open interval]a,b[.

29.8 RELATION BETWEEN THE SIGN OF THE DERIVATIVE AND MONOTONICITY OF FUNCTION

Consider a function whose curve is shown in the Fig. 29.5

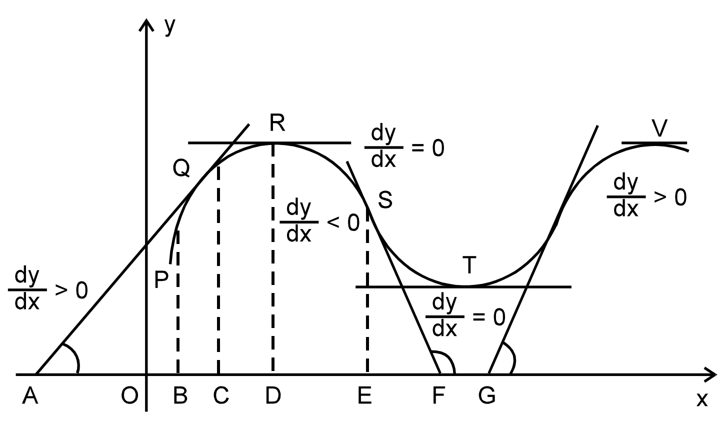


Fig. 29.5

We divide, our study of relation between sign of derivative of a function and its increasing or decreasing nature (monotonicity) into various parts as per Fig. 29.5

(i) P to R (ii) R to T (iii) T to V

(i) We observe that the ordinate (y-coordinate) for every succeeding point of the curve from P to R increases as also its x-coordinate. If (x_2, y_2) are the coordinates of a point that succeeds (x_1, y_1) then $x_2 > x_1$ yields $y_2 > y_1$ or $f(x_2) > f(x_1)$.

Also the tangent at every point of the curve between P and R makes acute angle with the positive direction of x-axis and thus the slope of the tangent at such points of the curve (except at R) is positive. At R where the ordinate is maximum the tangent is parallel to x-axis, as a result the slope of the tangent at R is zero.

We conclude for this part of the curve that

(a) The function is monotonically increasing from P to R.

(b) The tangent at every point (except at R) makes an acute angle with positive direction of x-axis.

(c) The slope of tangent is positive i.e. $\frac{dy}{dx} > 0$ for all points of the curve for which y is increasing.

(d) The slope of tangent at R is zero i.e. $\frac{dy}{dx} = 0$ where y is maximum.

(ii) The ordinate for every point between R and T of the curve decreases though its x-coordinate increases. Thus, for any point $x_2 > x_1$ yelds $y_2 < y_1$, or $f(x_2) < f(x_1)$.

Also the tangent at every point succeeding R along the curve makes obtuse angle with positive direction of x-axis. Consequently, the slope of the tangent is negative for all such points whose ordinate is decreasing. At T the ordinate attains minimum value and the tangent is parallel to x-axis and as a result the slope of the tangent at T is zero.

We now conclude :

(a) The function is monotonically decreasing from Rto T.

(b) The tangent at every point, except at T, makes obtuse angle with positive direction of x-axis.

(c) The slope of the tangent is negative i.e., $\frac{dy}{dx} < 0$ for all points of the curve for which y is decreasing.

(d) The slope of the tangent at T is zero i.e. $\frac{dy}{dx} = 0$ where the ordinate is minimum.

(iii) Again, for every point from T to V

The ordinate is constantly increasing, the tangent at every point of the curve between T and V makes acute angle with positive direction of x-axis. As a result of which the slope of the tangent at each of such points of the curve is positive.

Conclusively,

$$\frac{dy}{dx} > 0$$

at all such points of the curve except at T and V, where $\frac{dy}{dx} = 0$. The derivative $\frac{dy}{dx} < 0$ on one side, $\frac{dy}{dx} > 0$ on the other side of points R, T and V of the curve where $\frac{dy}{dx} = 0$.

Example 29.26 Find for what values of x, the function

$$f(x) = x^2 - 6x + 8$$

is increasing and for what values of x it is decreasing.

Solution : $f(x) = x^2 - 6x + 8$

$$f'(x) = 2x - 6$$

For $f(x)$ to be increasing, $f'(x) > 0$

i.e., $2x - 6 > 0$ or, $2(x - 3) > 0$

or, $x - 3 > 0$ or, $x > 3$

The function increases for x>3.

For $f(x)$ to be decreasing,

$$f'(x) < 0$$

i.e., $2x - 6 < 0$ or, $x - 3 < 0$

or, $x < 3$

Thus, the function decreases for $x < 3$.

Example 29.27 Find the interval in which $f(x) = 2x^3 - 3x^2 - 12x + 6$ is increasing or decreasing.

Solution : $f(x) = 2x^3 - 3x^2 - 12x + 6$

$$f'(x) = 6x^2 - 6x - 12$$

$$= 6\left(x^2 - x - 2\right)$$

$$= 6(x - 2)(x + 1)$$

For $f(x)$ to be increasing function of x,

$$f'(x) > 0$$

i.e. $6(x - 2)(x + 1) > 0$ or, $(x - 2)(x + 1) > 0$

Since the product of two factors is positive, this implies either both are positive or both are negative.

Either $x-2>0$ and $x+1>0$

i.e. $x>2$ and $x>-1$

$x>2$ implies $x>-1$

$\therefore$ $x>2$

or $x-2<0$ and $x+1<0$

i.e. $x<2$ and $x<-1$

$x<-1$ implies $x<2$

$\therefore$ $x<-1$

Hence f (x) is increasing for $x>2$ or $x<-1$.

Now, for f (x) to be decreasing,

$$f'(x)<0$$

or, $6(x-2)(x+1)<0$ or, $(x-2)(x+1)<0$

Since the product of two factors is negative, only one of them can be negative, the other positive.

Therefore,

Either

$x-2>0$ and $x+1<0$

i.e. $x>2$ and $x<-1$

There is no such possibility that $x>2$ and at the same time $x<-1$

or

$x-2<0$ and $x+1>0$

i.e. $x<2$ and $x>-1$

This can be put in this form

$-1<x<2$

$\therefore$ The function is decreasing in $-1<x<2$.

Example 29.28 Determine the intervals for which the function

$$f(x)=\frac{x}{x^2+1} \text{ is increasing or decreasing.}$$

Solution :
$$f'(x)=\frac{\left(x^2+1\right)\frac{dx}{dx}-x\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$=\frac{\left(x^2+1\right)-x\cdot(2x)}{\left(x^2+1\right)^2}$$

$$=\frac{1-x^2}{\left(x^2+1\right)^2}$$

$\therefore$
$$f'x=\frac{(1-x)(1+x)}{\left(x^2+1\right)^2}$$

As $\left(x^2+1\right)^2$ is positive for all real x.

Therefore, if $-1 < x < 0, (1-x)$ is positive and $(1+x)$ is positive, so $f'(x) > 0;$

$\therefore$ If $\quad 0 < x < 1, (1-x)$ is positive and $(1+x)$ is positive, so f'$f'(x) > 0;$

If $\quad x < -1, (1-x)$ is positive and $(1+x)$ is negative, so $f'(x) < 0;$

$\quad x > 1, (1-x)$ is negative and $(1+x)$ is positive, so $f'(x) < 0;$

Thus we conclude that

the function is increasing $\quad$ for $-1 < x < 0$ and $0 < x < 1$

or, $\quad$ for $-1 < x < 1$

and the function is decreasing for $x < -1$ or $x > 1$

Note : Points where $f'(x) = 0$ are critical points. Here, critical points are $x = -1$, $x = 1$.

Example 29.29 Show that

(a) $\quad f(x) = \cos x$ is decreasing in the interval $0 \le x \le \pi$.

(b) $\quad f(x) = x - \cos x$ is increasing for all x.

Solution :(a) $\quad f(x) = \cos x$

$$f'(x) = -\sin x$$

$f(x)$ is decreasing

If $\quad f'(x) < 0$

i.e., $\quad -\sin x < 0$

i.e., $\quad \sin x > 0$

sin x is positive in the first quadrant and in the second quadrant, therefore, sin x is positive in $0 \le x \le \pi$

$\therefore \quad f(x)$ is decreasing in $0 \le x \le \pi$

(b) $\quad f(x) = x - \cos x$

$$f'(x) = 1 - (-\sin x)$$

$$= 1 + \sin x$$

Now, we know that the minimum value of sinx is –1 and its maximum; value is 1 i.e.,sin x lies between –1 and 1 for all x,

i.e., $\quad -1 \le \sin x \le 1 \quad$ or $\quad 1 - 1 \le 1 + \sin x \le 1 + 1$

or $\quad 0 \le 1 + \sin x \le 2$

or $\quad 0 \le f'(x) \le 2$

or $\quad 0 \le f'(x)$

or $$f'(x) \geq 0$$

$\Rightarrow f(x) = x - \cos x$ is increasing for all x.

CHECK YOUR PROGRESS 29.8

Find the intervals for which the followiong functions are increasing or decreasing.

1. (a) $f(x) = x^2 - 7x + 10$ (b) $f(x) = 3x^2 - 15x + 10$

2. (a) $f(x) = x^3 - 6x^2 - 36x + 7$ (b) $f(x) = x^3 - 9x^2 + 24x + 12$

3. (a) $y = -3x^2 - 12x + 8$ (b) $f(x) = 1 - 12x - 9x^2 - 2x^3$

4. (a) $y = \frac{x-2}{x+1}, x \neq -1$ (b) $y = \frac{x^2}{x-1}, x \neq 1$ (c) $y = \frac{x}{2} + \frac{2}{x}, x \neq 0$

5. (a) Prove that the function log sin x is decreasing in $\left[\frac{\pi}{2}, \pi\right]$

   (b) Prove that the function cos x is increasing in the interval $[\pi, 2\pi]$

   (c) Find the intervals in which the function $\cos\left(2x + \frac{\pi}{4}\right), 0 \leq x \leq \pi$ is decreasing or increasing.

Find also the points on the graph of the function at which the tangents are parallel to x-axis.

29.9 MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We have seen the graph of a continuous function. It increases and decreases alternatively. If the value of a continious function increases upto a certain point then begins to decrease, then this point is called point of maximum and corresponding value at that point is called maximum value of the function. A stage comes when it again changes from decreasing to increasing . If the value of a continuous function decreases to a certain point and then begins to increase, then value at that point is called minimum value of the function and the point is called point of minimum.

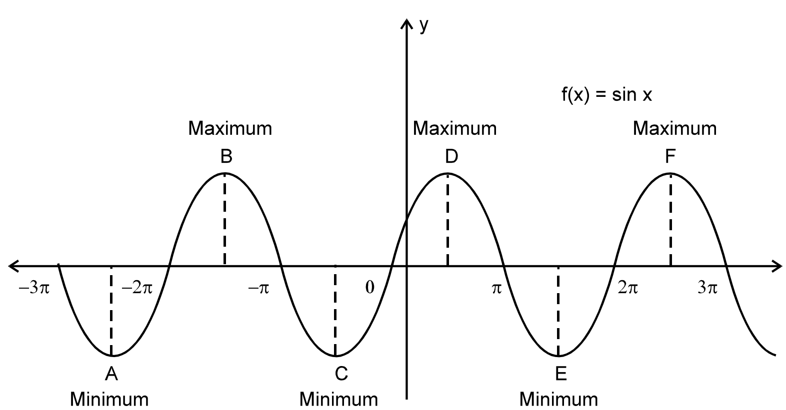


Fig. 29.6

Fig. 29.6 shows that a function may have more than one maximum or minimum values. So, for continuous function we have maximum (minimum) value in an interval and these values are not absolute maximum (minimum) of the function. For this reason, we sometimes call them as local maxima or local minima.

A function f (x) is said to have a maximum or a local maximum at the point x = a where a – b < a < a + b (See Fig. 29.7), if $f(a) \geq f(a \pm b)$ for all sufficiently small positive b.

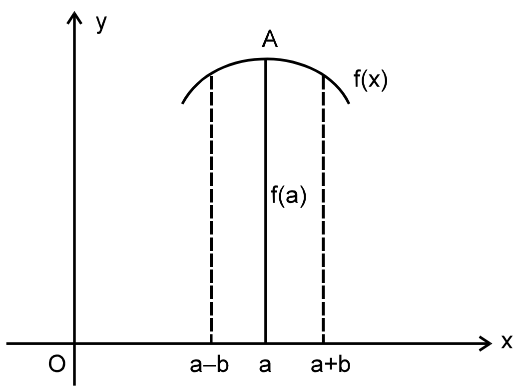


Fig. 29.7

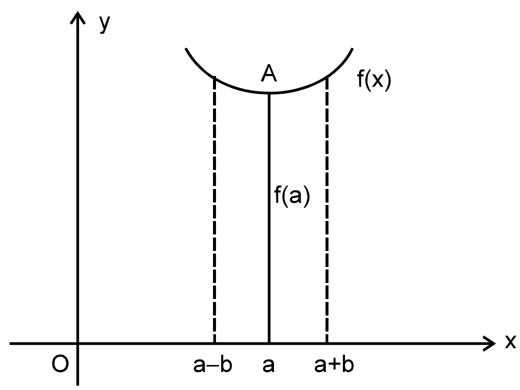


Fig. 29.8

A maximum (or local maximum) value of a function is the one which is greater than all other values on either side of the point in the immediate neighbourhood of the point.

A function f (x) is said to have a minimum (or local minimum) at the point x = a if $f(a) \leq f(a \pm b)$ where a – b< a < a + b

for all sufficiently small positive b.

In Fig. 25.8, the function f(x) has local minimum at the point x = a.

A minimum (or local miunimum) value of a function is the one which is less than all other values, on either side of the point in the immediate neighbourhood of the point.

Note : A neighbourhood of a point $x \in R$ is defined by open internal $]x-\in[$, when $\in>0$.

29.9.1 CONDITIONS FOR MAXIMUM OR MINIMUM

We know that derivative of a function is positive when the function is increasing and the derivative is negative when the function is decreasing. We shall apply this result to find the condition for maximum or a function to have a minimum. Refer to Fig. 25.6, points B,D, F are points of maxima and points A,C,E are points of minima.

Now, on the left of B, the function is increasing and so $f'(x) > 0$, but on the right of B, the function is decreasing and, therefore, $f'(x) < 0$. This can be achieved only when f '(x) becomes zero somewhere in betwen. We can rewrite this as follows :

A function f (x) has a maximum value at a point if (i) f '(x) = 0 and (ii) f '(x) changes sign from positive to negative in the neighbourhood of the point at which f '(x)=0 (points taken from left to right).

Now, on the left of C (See Fig. 29.6), function is decreasing and f '(x) therefore, is negative and on the right of C, f (x) is increasing and so f '(x) is positive. Once again f '(x) will be zero before having positive values. We rewrite this as follows :

A function f (x) has a minimum value at a point if (i) f '(x)=0, and (ii) f '(x) changes sign from negative to positive in the neighbourhood of the point at which f '(x) = 0.

We should note here that f '(x) = 0 is necessary condition and is not a sufficient condition for maxima or minima to exist. We can have a function which is increasing, then constant and then again increasing function. In this case, f '(x) does not change sign. The value for which f '(x) = 0 is not a point of maxima or minima. Such point is called point of inflexion.

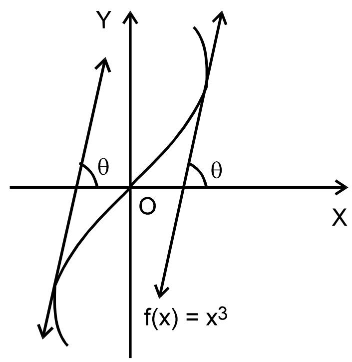


Fig. 29.9

For example, for the function $f(x) = x^3, x = 0$ is the point of inflexion as $f'(x) = 3x^2$ does not change sign as x passes through 0. f'(x) is positive on both sides of the value '0' (tangents make acute angles with x-axis) (See Fig. 29.9).

Hence $f(x) = x^3$ has a point of inflexion at x = 0.

The points where f '(x) = 0 are called stationary points as the rate ofchange of the function is zero there. Thus points of maxima and minima are stationary points.

Remarks

The stationary points at which the function attains either local maximum or local minimum values are also called extreme points and both local maximum and local minimum values are called extreme values of f (x). Thus a function attains an extreme value at x=a if f (a) is either a local maximum or a local minimum.

29.9.2 METHOD OF FINDING MAXIMA OR MINIMA

We have arrived at the method of finding the maxima or minima of a function. It is as follows :

(i) Find f'(x)

(ii) Put f '(x)=0 and find stationary points

(iii) Consider the sign of f '(x) in the neighbourhood of stationary points. If it changes sign from +ve to –ve, then f (x) has maximum value at that point and if f '(x) changes sign from –ve to +ve, then f (x) has minimum value at that point.

(iv) If f '(x) does not change sign in the neighbourhood of a point then it is a point of inflexion.

Example 29.30 Find the maximum (local maximum) and minimum (local minimum) points of the function $f(x) = x^3 - 3x^2 - 9x$.

Solution : Here

$$f(x) = x^3 - 3x^2 - 9x$$

$$f'(x) = 3x^2 - 6x - 9$$

Step I. Now $\quad f'(x)=0$ gives us $3x^2-6x-9=0$

or $\quad x^2-2x-3=0$

or $\quad (x-3)(x+1)=0$

or $\quad x=3,-1$

$\therefore$ Stationary points are $\quad x=3,\ x=-1$

Step II. At $\quad x=3$

For $\quad x<3 \quad f'(x)<0$

and for $\quad x>3 \quad f'(x)>0$

$\therefore$ f '(x) changes sign from –ve to +ve in the neighbourhood of 3.

$\therefore$ f (x) has minimum value at x = 3.

Step III. At $\quad x=-1,$

For $\quad x<-1, \quad f'(x)>0$

and for $\quad x>-1, \quad f'(x)<0$

$\therefore$ f '(x) changes sign from +ve to –ve in the neighbourhood of –1.

$\therefore$ f(x) has maximum value at x=–1.

$\therefore$ x = –1 and x = 3 give us points of maxima and minima respectively. If we want to find maximum value (minimum value), then we have

$$\text{maximum value }=f(-1)=(-1)^3-3(-1)^2-9(-1)$$

$$=-1-3+9=5$$

and $\quad \text{minimum value }=f(3)=3^3-3(3)^2-9(3)=-27$

$\therefore$ $(-1,5)$ and $(3,-27)$ are points of local maxima and local minima respectively.

Example 29.31 Find the local maximum and the local minimum of the function

$$f(x)=x^2-4x$$

Solution : $\quad f(x)=x^2-4x$

$\therefore \quad f'(x)=2x-4=2(x-2)$

Putting f ' (x)=0 yields 2x–4=0, i.e., x=2.

We have to examine whether x = 2 is the point of local maximum or local minimum or neither maximum nor minimum.

Let us take x = 1.9 which is to the left of 2 and x = 2.1 which is to the right of 2 and find f'(x) at these points.

$$f'(1.9) = 2(1.9-2) < 0$$

$$f'(2.1) = 2(2.1-2) > 0$$

Since $f'(x) < 0$ as we approach 2 from the left and $f'(x) > 0$ as we approach 2 from the right, therefore, there is a local minimum at $x = 2$.

We can put our findings for sign of derivatives of f(x) in any tabular form including the one given below :

sign of $f'(x)$

point $x = 2$

left of 2	right of 2
$f'(x) < 0$	$f'(x) < 0$

Local minimum

Example 29.32 Find all local maxima and local minima of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 8$$

Solution : $$f(x) = 2x^3 - 3x^2 - 12x + 8$$

$\therefore$ $$f'(x) = 6x^2 - 6x - 12$$

$$= 6(x^2 - x - 2)$$

$\therefore$ $$f'(x) = 6(x+1)(x-2)$$

Now solving f '(x)=0 for x, we get

$$6(x+1)(x-2) = 0$$

$\Rightarrow$ $$x = -1, 2$$

Thus $$f'(x) = 0 \text{ at } x = -1, 2.$$

We examine whether these points are points of local maximum or local minimum or neither of them.

Consider the point $x = -1$

Let us take $x = -1.1$ which is to the left of -1 and $x = -0.9$ which is to the right of -1 and find $f'(x)$ at these points.

$$f'(-1.1) = 6(-1.1+1)(-1.1-2)\text{, which is positive i.e. } f'(x)>0$$

$$f'(-0.9) = 6(-0.9+1)(-0.9-2)\text{, which is negative i.e. } f'(x)<0$$

Thus, at $x = -1$, there is a local maximum.

Consider the point $x = 2$.

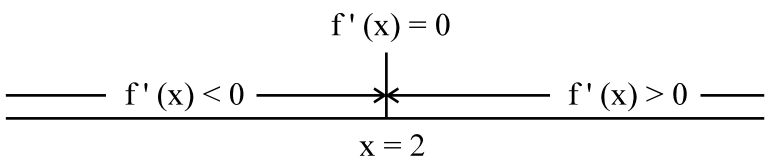
Now, let us take $x = 1.9$ which is to the left of $x = 2$ and $x = 2.1$ which is to the right of $x = 2$ and find $f'(x)$ at these points.

$$f'(1.9) = 6(1.9+1)(1.9-2)$$
$$= 6 \times (\text{Positive number}) \times (\text{negative number})$$
$$= \text{a negative number}$$

i.e. $\quad f'(1.9) < 0$

and $\quad f'(2.1)=6(2.1+1)(2.1-2)$, which is positive

i.e., $\quad f'(2.1)>0$



$\because \quad f'(x) < 0$ as we approach 2 from the left

and $\quad f'(x) > 0$ as we approach 2 from the right.

$\therefore \quad x = 2$ is the point of local minimum

Thus $f(x)$ has local maximum at $x = -1$, maximum value of $f(x) = -2-3+12+8 = 15$

$f(x)$ has local minimum at $x = 2$, minimum value of $f(x) = 2(8)-3(4)-12(2)+8 = -12$

Sign of $f'(x)$

Point $x = -1$			Point $x = 2$	
Left of -1	Right of -1		Left of 2	Right of 2
positive	negative		negative	positive
local maximum			local minimum	

Example 29.33 Find local maximum and local minimum, if any, of the following function

$$f(x) = \frac{x}{1+x^2}$$

Solution :
$$f(x) = \frac{x}{1+x^2}$$

Then $$f'(x) = \frac{(1+x^2)1-(2x)x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

For finding points of local maximum or local minimum, equate f '(x) to 0.

i.e. $$\frac{1-x^2}{(1+x^2)^2} = 0$$

$\Rightarrow$ $$1-x^2 = 0$$

or $(1+x)(1-x) = 0$ or $x = 1, -1$.

Consider the value x = 1.

The sign of f '(x) for values of x slightly less than 1 and slightly greater than 1 changes from positive to negative. Therefore there is a local maximum at x =1, and the local maximum value $= \dfrac{1}{1+(1)^2} = \dfrac{1}{1+1} = \dfrac{1}{2}$

Now consider x = – 1.

f '(x) changes sign from negative to positive as x passes through –1, therefore, f (x) has a local minimum at x = – 1

Thus, the local minimum value $= \dfrac{-1}{2}$

Example 29.34 Find the local maximum and local minimum, if any, for the function

$$f(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$$

Solution : We have $f(x) = \sin x + \cos x$

$$f'(x) = \cos x - \sin x$$

For local maxima/minima, f '(x) = 0

$\therefore$ $$\cos x - \sin x = 0$$

or, $\tan x = 1$ or, $x = \dfrac{\pi}{4}$ in $0 < x < \dfrac{\pi}{2}$

At $x = \dfrac{\pi}{4}$,

For $x < \dfrac{\pi}{4}$, $\cos x > \sin x$

$\therefore$ $$f'(x) = \cos x - \sin x > 0$$

For $\quad x > \frac{\pi}{4}, \; \cos x - \sin x < 0$

$\therefore \quad f'(x) = \cos x - \sin x < 0$

$\therefore$ f '(x) changes sign from positive to negative in the neighbourhood of $\frac{\pi}{4}$.

$\therefore$ $x = \frac{\pi}{4}$ is a point of local maxima.

Maximum value $= f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$\therefore$ Point of local maxima is $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

CHECK YOUR PROGRESS 29.9

Find all points of local maxima and local minima of the following functions. Also , find the maxima and minima at such points.

1. $x^2 - 8x + 12$
2. $x^3 - 6x^2 + 9x + 15$
3. $2x^3 - 21x^2 + 36x - 20$
4. $x^4 - 62x^2 + 120x + 9$
5. $(x-1)(x-2)^2$
6. $\dfrac{x-1}{x^2+x+2}$

29.10 USE OF SECOND DERIVATIVE FOR DETERMINATION OF MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We now give below another method of finding local maximum or minimum of a function whose second derivative exists. Various steps used are :

(i) Let the given function be denoted by f (x).

(ii) Find f ' (x) and equate it to zero.

(iii) Solve f ' (x)=0, let one of its real roots be x = a.

(iv) Find its second derivative, f ''(x). For every real value 'a' of x obtained in step (iii), evaluate f'' (a). Then if

f ''(a) <0 then x =a is a point of local maximum.

f ''(a)>0 then x = a is a point of local minimum.

f ''(a)=0 then we use the sign of f '(x) on the left of 'a' and on the right of 'a' to arrive at the result.

Example 29.35 Find the local minimum of the following function :

$$2x^3 - 21x^2 + 36x - 20$$

Solution : Let $\quad f(x) = 2x^3 - 21x^2 + 36x - 20$

Then $\quad f'(x) = 6x^2 - 42x + 36$

$$= 6(x^2 - 7x + 6)$$

$$= 6(x-1)(x-6)$$

For local maximum or min imum

$$f'(x) = 0$$

or $\quad 6(x-1)(x-6) = 0 \quad \Rightarrow x = 1, 6$

Now $\quad f''(x) = \frac{d}{dx} f'(x)$

$$= \frac{d}{dx}\left[6(x^2 - 7x + 6)\right]$$

$$= 12x - 42$$

$$= 6(2x - 7)$$

For $\quad x = 1,\ f''(1) = 6(2.1 - 7) = -30 < 0$

$x = 1$ is a point of local maximum.

and $f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20 = -3$ is a local maximum.

For $x = 6$,

$f''(6) = 6(2.6 - 7) = 30 > 0$

$\therefore \quad x = 6$ is a point of local minimum

and $f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20 = -128$ is a local minimum.

Example 29.36 Find local maxima and minima (if any) for the function

$$f(x) = \cos 4x; \quad 0 < x < \frac{\pi}{2}$$

Solution : $\quad f(x) = \cos 4x$

$\therefore \quad f'(x) = -4\sin 4x$

Now, $\quad f'(x) = 0 \quad \Rightarrow \quad -4\sin 4x = 0$

or, $\quad \sin 4x = 0 \quad$ or, $\quad 4x = 0, \pi, 2\pi$

or, $$x = 0, \frac{\pi}{4}, \frac{\pi}{2}$$

$\therefore$ $$x = \frac{\pi}{4} \qquad \left[\because \ 0 < x < \frac{\pi}{2}\right]$$

Now, $$f''(x) = -16\cos 4x$$

at $$x = \frac{\pi}{4}, f''(x) = -16\cos\pi$$

$$= -16(-1) = 16 > 0$$

$\therefore$ f (x) is minimum at $x = \frac{\pi}{4}$

Minimum value $$f\left(\frac{\pi}{4}\right) = \cos\pi = -1$$

Example 29.37 (a) Find the maximum value of $2x^3 - 24x + 107$ in the interval [–3,–1].

(b) Find the minimum value of the above function in the interval [1,3].

Solution : Let $f(x) = 2x^3 - 24x + 107$

$$f'(x) = 6x^2 - 24$$

For local maximum or minimum,

$$f'(x) = 0$$

i.e. $$6x^2 - 24 = 0 \qquad \Rightarrow \qquad x = -2, 2$$

Out of two points obtained on solving f '(x)=0, only–2 belong to the interval [–3,–1]. We shall, therefore, find maximum if any at x=–2 only.

Now $$f''(x) = 12x$$

$\therefore$ $$f''(-2) = 12(-2) = -24$$

or $$f''(-2) < 0$$

which implies the function f (x) has a maximum at x =–2.

$\therefore$ Required maximum value $$= 2(-2)^3 - 24(-2) + 107$$

$$=139$$

Thus the point of maximum belonging to the given interval [–3,–1] is –2 and, the maximum value of the function is 139.

(b) Now $$f''(x) = 12x$$

MODULE - VIII
Calculus

Notes

$\therefore \quad f''(2) = 24 > 0, \quad [\because\ 2 \text{ lies in } [1, 3]]$

which implies, the function f (x) shall have a minimum at x =2.

$\therefore$ Required minimum $= 2(2)^3 - 24(2) + 107$

$=75$

Example 29.38 Find the maximum and minimum value of the function

$$f(x) = \sin x(1+\cos x) \text{ in } (0, \pi).$$

Solution : We have, $f(x) = \sin x(1+\cos x)$

$$f'(x) = \cos x(1+\cos x) + \sin x(-\sin x)$$

$$= \cos x + \cos^2 x - \sin^2 x$$

$$= \cos x + \cos^2 x - (1-\cos^2 x) = 2\cos^2 x + \cos x - 1$$

For stationary points, $f'(x) = 0$

$$\therefore \quad 2\cos^2 x + \cos x - 1 = 0$$

$$\therefore \quad \cos x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1, \frac{1}{2}$$

$$\therefore \quad x = \pi, \frac{\pi}{3}$$

Now, $f(0) = 0$

$$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

and $f(\pi) = 0$

$\therefore$ f (x) has maximum value $\frac{3\sqrt{3}}{4}$ at $x = \frac{\pi}{3}$

and miminum value 0 at x = 0 and $x = \pi$.

CHECK YOUR PROGRESS 29.10

Find local maximum and local minimum for each of the following functions using second order derivatives.

1. $2x^3 + 3x^2 - 36x + 10$
2. $-x^3 + 12x^2 - 5$

3. $(x-1)(x+2)^2$

4. $x^5 - 5x^4 + 5x^3 - 1$

5. $\sin x(1+\cos x), 0 < x < \frac{\pi}{2}$

6. $\sin x + \cos x, 0 < x < \frac{\pi}{2}$

7. $\sin 2x - x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

29.11 APPLICATIONS OF MAXIMA AND MINIMA TO PRACTICAL PROBLEMS

The application of derivative is a powerful tool for solving problems that call for minimising or maximising a function. In order to solve such problems, we follow the steps in the following order :

(i) Frame the function in terms of variables discussed in the data.

(ii) With the help of the given conditions, express the function in terms of a single variable.

(iii) Lastly, apply conditions of maxima or minima as discussed earlier.

Example 29.39 Find two positive real numbers whose sum is 70 and their product is maximum.

Solution : Let one number be x. As their sum is 70, the other number is 70–x. As the two numbers are positive, we have, $x > 0, 70 - x > 0$

$$70 - x > 0 \quad \Rightarrow \quad x < 70$$

$$\therefore \quad 0 < x < 70$$

Let their product be f (x)

Then $\quad f(x) = x(70 - x) = 70x - x^2$

We have to maximize the prouct f (x).

We, therefore, find f '(x) and put that equal to zero.

$$f'(x) = 70 - 2x$$

For maximum product, $f'(x) = 0$

or $\quad 70 - 2x = 0$

or $\quad x = 35$

Now $f''(x) = -2$ which is negative. Hence f (x) is maximum at x = 35

The other number is $70 - x = 35$

Hence the required numbers are 35, 35.

Example 29.40 Show that among rectangles of given area, the square has the least perimeter.

Solution : Let x, y be the length and breadth of the rectangle respectively.

$\therefore$ Its area $= xy$

Since its area is given, represent it by A, so that we have

$$A = xy$$

or $$y = \frac{A}{x} \quad \text{... (i)}$$

Now, perimeter say P of the rectangle = 2 (x + y)

or $$P = 2\left(x + \frac{A}{x}\right)$$

$\therefore$ $$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right) \quad \text{...(ii)}$$

For a minimum $P, \frac{dP}{dx} = 0.$

i.e. $$2\left(1 - \frac{A}{x^2}\right) = 0$$

or $$A = x^2 \quad \text{or} \quad \sqrt{A} = x$$

Now, $$\frac{d^2P}{dx^2} = \frac{4A}{x^3}, \text{ which is positive.}$$

Hence perimeter is minimum when $x = \sqrt{A}$

$\therefore$ $$y = \frac{A}{x}$$

$$= \frac{x^2}{x} = x \qquad (\because A = x^2)$$

Thus, the perimeter is minimum when rectangle is a square.

Example 29.41 An open box with a square base is to be made out of a given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

Solution : Let x be the side of the square base of the box and y its height.

Total surface area of othe box $= x^2 + 4xy$

$\therefore$ $$x^2 + 4xy = a^2 \quad \text{or} \quad y = \frac{a^2 - x^2}{4x}$$

Volume of the box, $V = \text{base area} \times \text{height}$

$$= x^2 y = x^2\left(\frac{a^2 - x^2}{4x}\right)$$

or $$V = \frac{1}{4}\left(a^2 x - x^3\right) \qquad \text{...(i)}$$

$\therefore$ $$\frac{dV}{dx} = \frac{1}{4}\left(a^2 - 3x^2\right)$$

For maxima/minima $\frac{dV}{dx} = 0$

$\therefore$ $$\frac{1}{4}\left(a^2 - 3x^2\right) = 0$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\text{Volume} = \frac{1}{4}\left(\frac{\left(a^3\right)}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right) = \frac{a^3}{6\sqrt{3}} \qquad \text{..(iii)}$$

Again $$\frac{d^2V}{dx^2} = \frac{d}{dx}\frac{1}{4}\left(a^2 - 3x^2\right) = -\frac{3}{2}x$$

x being the length of the side, is positive.

$\therefore$ $$\frac{d^2V}{dx^2} < 0$$

$\therefore$ The volume is maximum.

Hence maximum volume of the box $= \frac{a^3}{6\sqrt{3}}$.

Example 29.42 Show that of all rectangles inscribed in a given circle, the square has the maximum area.

Solution : Let ABCD be a rectangle inscribed in a circle of radius r. Then diameter AC= 2r

Let $\qquad AB = x$ and $BC = y$

Then $\qquad AB^2 + BC^2 = AC^2 \qquad$ or $\qquad x^2 + y^2 = (2r)^2 = 4r^2$

Now area A of the rectangle = xy

$\therefore$ $$A = x\sqrt{4r^2 - x^2}$$

$\therefore$ $$\frac{dA}{dx} = \frac{x(-2x)}{2\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} \cdot 1$$

$$= \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

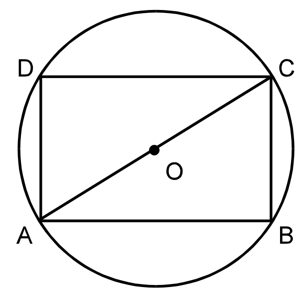


Fig. 29.10

For maxima/minima, $\frac{dA}{dx} = 0$

$$\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0 \Rightarrow x = \sqrt{2}r$$

Now $$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}\,(-4x) - \left(4r^2 - 2x^2\right)\frac{(-2x)}{2\sqrt{4r^2 - x^2}}}{\left(4r^2 - x^2\right)}$$

$$= \frac{-4x\left(4r^2 - x^2\right) + x\left(4r^2 - 2x^2\right)}{\left(4r^2 - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-4\sqrt{2}\left(2r^2\right) + 0}{\left(2r^2\right)^{\frac{3}{2}}}$$ ... (Putting $x = \sqrt{2}r$)

$$= \frac{-8\sqrt{2}r^3}{2\sqrt{2}r^3} = -4 < 0$$

Thus, A is maximum when $x = \sqrt{2}r$

Now, from (i), $y = \sqrt{4r^2 - 2r^2} = \sqrt{2}\ r$

x = y. Hence, rectangle ABCD is a square.

Example 29.43 Show that the height of a closed right circular cylinder of a given volume andleast surface is equal to its diameter.

Solution : Let V be the volume, r the radius and h the height of the cylinder.

Then, $$V = \pi r^2 h$$

or $$h = \frac{V}{\pi r^2}$$ ... (i)

Now surface area $$S = 2\pi rh + 2\pi r^2$$

$$= 2\pi r.\frac{V}{\pi r^2} + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2$$

Now $$\frac{dS}{dr} = \frac{-2V}{r^2} + 4\pi r$$

For minimum surface area, $\frac{dS}{dr} = 0$

$\therefore$ $$\frac{-2V}{r^2} + 4\pi r = 0$$

or $$V = 2\pi r^3$$

From (i) and (ii), we get $$h = \frac{2\pi r^3}{\pi r^2} = 2r \qquad \text{..(ii)}$$

Again, $$\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 4\pi = 8\pi + 4\pi \qquad \text{... [Using (ii)]}$$

$$= 12\pi > 0$$

$\therefore$ S is least when h = 2r

Thus, height of the cylidner = diameter of the cylinder.

Example 29.44 Show that a closed right circular cylinder of given surface has maximum volume if its height equals the diameter of its base.

Solution : Let S and V denote the surface area and the volume of the closed right circular cylinder of height h and base radius r.

Then $$S = 2\pi rh + 2\pi r^2 \qquad \text{.....(i)}$$

(Here surface is a constant quantity, being given)

$$V = \pi r^2 h$$

$\therefore$ $$V = \pi r^2 \left[\frac{S - 2\pi r^2}{2\pi r}\right]$$

$$= \frac{r}{2}\left[S - 2\pi r^2\right]$$

$$V = \frac{Sr}{2} - \pi r^3$$

$$\frac{dV}{dr} = \frac{S}{2} - \pi\left(3r^2\right)$$

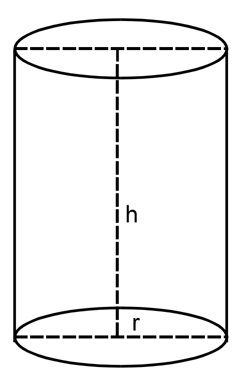


Fig. 29.11

For maximum or minimum, $\frac{dV}{dr} = 0$

i.e., $$\frac{S}{2} - \pi\left(3r^2\right) = 0$$

or $$S = 6\pi r^2$$

From (i), we have $$6\pi r^2 = 2\pi rh + 2\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 = 2\pi rh$$

$$\Rightarrow \quad 2r = h \qquad ...(ii)$$

Also, $$\frac{d^2V}{dr^2} = \frac{d}{dr}\left[\frac{S}{2} - 3\pi r^2\right]$$

$$= -6\pi r, \qquad \because \frac{d}{dr}\left(\frac{S}{2}\right) = 0$$

= a negative quantity

Hence the volume of the right circular cylinder is maximum when its height is equal to twice its radius i.e. when h = 2r.

Example 29.45 A square metal sheet of side 48 cm. has four equal squares removed from the corners and the sides are then turned up so as to form an open box. Determine the size of the square cut so that volume of the box is maximum.

Solution : Let the side of each of the small squares cut be x cm, so that each side of the box to be made is (48–2x) cm. and height x cm.

Now $x > 0$, $48 - 2x > 0$, i.e. $x < 24$

$\therefore$ x lies between 0 and 24 or $0 < x < 24$

Now, Volume V of the box

$$= (48 - 2x)(48 - 2x)x$$

i.e. $$V = (48 - 2x)^2 \cdot x$$

$$\therefore \quad \frac{dV}{dx} = (48 - 2x)^2 + 2(48 - 2x)(-2)x$$

$$= (48 - 2x)(48 - 6x)$$

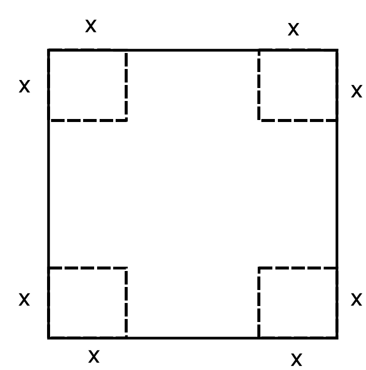


Fig. 29.12

Condition for maximum or minimum is $\frac{dV}{dx} = 0$

i.e., $\left(48-2x\right)\left(48-6x\right)=0$

We have either $x=24$, or $x=8$

$\because$ $0<x<24$

$\therefore$ Rejecting x = 24, we have, x = 8 cm.

Now, $$\frac{d^2V}{dx^2}=24x-384$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=8}=192-384=-192<0,$$

Hence for x = 8, the volume is maximum.

Hence the square of side 8 cm. should be cut from each corner.

Example 29.46 The profit function P (x) of a firm, selling x items per day is given by

$$P(x)=\left(150-x\right)x-1625.$$

Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.

Solution : It is given that 'x' is the number of items produced and sold out by the firm every day. In order to maximize profit,

$$P'\left(x\right)=0 \text{ i.e. } \frac{dP}{dx}=0$$

or $$\frac{d}{dx}\left[\left(150-x\right)x-1625\right]=0$$

or $$150-2x=0$$

or $$x=75$$

Now, $\frac{d}{dx}P'\left(x\right)=P''\left(x\right)=-2$ = a negative quantity. Hence P (x) is maximum for x = 75.

Thus, the firm should manufacture only 75 items a day to make maximum profit.

Now, Maximum Profit $=P\left(75\right)=\left(150-75\right)75-1625$

= Rs. (75×75–1625)

= Rs. (5625–1625)

= Rs. 4000

Example 29.47 Find the volume of the largest cylinder that can be inscribed in a sphere of radius 'r' cm.

Solution : Let h be the height and R the radius of the base of the inscribed cylinder. Let V be the volume of the cylinder.

Then $$V = \pi R^2 h \qquad ...(i)$$

From Δ OCB, we have

$$r^2 = \left(\frac{h}{2}\right)^2 + R^2 \qquad ...\left(\because OB^2 = OC^2 + BC^2\right)$$

$$\therefore \quad R^2 = r^2 - \frac{h^2}{4} \qquad ...(ii)$$

Now $$V = \pi\left(r^2 - \frac{h^2}{4}\right)h = \pi r^2 h - \pi\frac{h^3}{4}$$

$$\therefore \quad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}$$

For maxima/minima, $\frac{dV}{dh} = 0$

$$\therefore \quad \pi r^2 - \frac{3\pi h^2}{4} = 0$$

$$\Rightarrow \quad h^2 = \frac{4r^2}{3} \quad \Rightarrow \quad h = \frac{2r}{\sqrt{3}}$$

Now $$\frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$$

$$\therefore \quad \frac{d^2V}{dh^2}\left(\text{at } h = \frac{2r}{\sqrt{3}}\right) = -\frac{3\pi \times 2r}{2 \times \sqrt{3}}$$

$$= -\sqrt{3}\pi r < 0$$

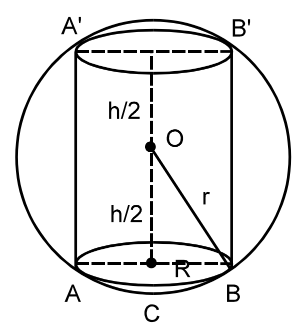


Fig. 29.13

$$\therefore \quad V \text{ is maximum at } h = \frac{2r}{\sqrt{3}}$$

Putting $h = \frac{2r}{\sqrt{3}}$ in (ii), we get

$$R^2 = r^2 - \frac{4r^2}{4 \times 3} = \frac{2r^2}{3}, \quad \therefore \quad R = \sqrt{\frac{2}{3}}r$$

Maximum volume of the cylinder $= \pi R^2 h$

$$= \pi \cdot \left(\frac{2}{3}r^2\right)\frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}\ cm^3.$$

CHECK YOUR PROGRESS 29.11

1. Find two numbers whose sum is 15 and the square of one multiplied by the cube of the other is maximum.

2. Divide 15 into two parts such that the sum of their squares is minimum.

3. Show that among the rectangles of given perimeter, the square has the greatest area.

4. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

5. A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter be 30 m, find the dimensions so that the greatest possible amount of light may be admitted.

6. Find the radius of a closed right circular cylinder of volume 100 c.c. which has the minimum total surface area.

7. A right circular cylinder is to be made so that the sum of its radius and its height is 6 m. Find the maximum volume of the cylinder.

8. Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third that of the cone.

9. A conical tent of the given capacity (volume) has to be constructed. Find the ratio of the height to the radius of the base so as to minimise the canvas requried for the tent.

10. A manufacturer needs a container that is right circular cylinder with a volume 16π cubic meters. Determine the dimensions of the container that uses the least amount of surface (sheet) material.

11. A movie theatre's management is considering reducing the price of tickets from Rs.55 in order to get more customers. After checking out various facts they decide that the average number of customers per day 'q' is given by the function where x is the amount of ticket price reduced. Find the ticket price othat result in maximum revenue.

$$q = 500+100\,x$$

where x is the amount of ticket price reduced. Find the ticket price that result is maximum revenue.

LET US SUM UP

- The equation of tangent at (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = [f'(x)]_{at(x_1,y_1)} \{x - x_1\}$$

- The equation of normal at (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = \left[\frac{-1}{f'(x)}\right]_{(x_1,y_1)} (x - x_1)$$

- The equation of tangent to a curve $y = f(x)$ at (x_1, y_1) and parallel to x-axis is given by $y = y_1$ and parallel to y-axis is given by $x = x_1$

- $y = f(x)$ be a function of x.

  The rate of change of y per unit change in x

  $$= \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

  $\frac{dy}{dx}$ represent the rate of change of y w.r. to x.

  If $y = f(t)$ and $x = g(t)$

  So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0$

- $\Delta y = \frac{dy}{dx}\Delta x + \varepsilon.\Delta x$

  $\because$ $\varepsilon.\Delta x$ is a very-very small quantity that can be neglected, therefore

  $$\Delta y = \frac{dy}{dx}.\Delta x \text{, approximately}$$

- **Increasing function :** A function f (x) is said to be increasing in the closed interval [a,b]

  if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$

- **Decreasing function :** A function f (x) is said to be decreasing in the closed interval [a,b]

  if $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$

- f (x) is increasing in an open interval]a,b[

  if $f'(x) > 0$ for all $x \in [a, b]$

- f (x) is decreasing in an open interval]a,b[

  if $f'(x) < 0$ for all $x \in [a, b]$

- **Monotonic function :**
  (i) A function is said to be monotonic (increasing) if it increases in the given interval.
  (ii) A function f (x) is said to be monotonic (decreasing) if it decreases in the given interval.

  A function f (x) which increases and decreases in a given interval, is not monotonic.
- In an interval around the point x = a of the function f (x),
  (i) if f ' (x) > 0 on the left of the point 'a' and f '(x) < 0 on the right of the point x = a, then f (x) has a local maximum.
  (ii) if f '(x) <0 on the left of the point 'a' and f ' (x) >0 on the right of the point x = a, then f (x) has a local minimum.
- If f (x) has a local maximum or local minimum at x = a and f (x) is derivable at x = a, then f '(a) = 0
- (i) If f '(x) changes sign from positive to negative as x passes through 'a', then f (x) has a local maximum at x = a.
  (ii) If f ' (x) changes sign from negative to positive as x passes othrough 'a', then f (x) has a local minimum at x = a.
- **Second order derivative Test :**
  (i) If f '(a) = 0, and f ''(a) < 0; then f (x) has a local maximum at x = a.
  (ii) If f '(a) = 0, and f ''(a) >0; then f (x) has a local minimum at x = a.
  (iii) In case f '(a) = 0, and f '' (a) = 0; then to determine maximum or minimum at x = a, we use the method of change of sign of f '(x) as x passes through 'a' to.

SUPPORTIVE WEBSITES

http://www.youtub.com/watch?v=IDY9JcFaRd4

http://www.youtub.com/watch?v=bGNMXfaNR5Q

http://mathworld.wolfram.com/PartialDerivative.html

http://en.wikipedia.org/wiki/Partial_derivative

http://en.wikipedia.org/wiki/Integral

TERMINAL EXERCISE

1. The side of a square is increasing at the rate of 0.2 cm/sec. Find the rate of increase of the perimeter of the square.
2. The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?
3. A man is walking at the rate of 4.5 km/hr towards the foot of a tower 120 m high. At what rate is he approching the top of the tower when he is 50 m away from the tower?

4. Sand is pouring from a pipe at the rate of 12 cm^3/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand-cone increasing when the height is 4 cm?

5. A man 2 metres high, walks at a uniform speed of 6 metres per minute away from a lamp past, 5 metres high. Find the rate at which the length of his shadow increases.

6. A particle moves along the curve, $y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

7. A stone is dropped into a quiet lake and waves move in a circle at a speed of 3.5 cm/sec. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing?

8. A stone dropped into a still pond produces a series of continually enlarging concentric circles. Find the rate at which one of them the enlarging when its diameter is 12 cm assuming the wave is then recording from the centre at the rate of 3 cm/sec.

9. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.

10. A particle moves along the curve $y = \frac{2}{3}x^3 + 1$. Find the points on the curve at which the y, coordinate is changing twice as fast as x-coordinate.

11. The total revenue in rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue when 5 units of the product are sold.

12. The total cost C(x) associated with the production of x units of a product is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000.$$

Find the marginal cost when 3 units are produced.

Using differentials find the approximate value of the following (13 – 19)

13. $\sqrt{25.02}$ 14. $\sqrt{49.5}$

15. (i) $\sqrt{401}$ (ii) $\sqrt{0.24}$

16. (i) $\sqrt{0.0037}$ (ii) $(26)^{\frac{1}{3}}$

17. (i) $(66)^{\frac{1}{3}}$ (ii) $(82)^{\frac{1}{4}}$

18. (i) $(32.15)^{\frac{1}{5}}$ (ii) $(31.9)^{\frac{1}{5}}$

19. (i) $\frac{1}{(2.002)^2}$ (ii) $\frac{1}{\sqrt{25.1}}$

20. Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 15x + 5$

21. Find the approximate change in the volume of a cube of side x meter caused by increasing the side by 3%.

22. Find the approximate change in the surface area of a cube of sides x meters by decreasing the side by 1%.
23. Find the approximate change in the volume of a cube of side x meters caused by increasing the side by 1%.
24. Find the approximate value of $f(5.001)$, where
    $f(x) = x^3 - 7x^2 + 15$
25. Find the slopes of tangents and normals to each of the following curves at the indicated points :

    (i) $y = \sqrt{x}$ at $x=9$

    (ii) $y = x^3 + x$ at $x = 2$

    (iii) $x = a(\theta - \sin\theta),\ y = a(1+\cos\theta)$ at $\theta = \dfrac{\pi}{2}$

    (iv) $y = 2x^2 + \cos x$ at $x = 0$

    (v) $xy = 6$ at $(1,6)$
26. Find the equations of tangent and normal to the curve

    $$x = a\cos^3\theta,\ y = a\sin^3\theta \text{ at } \theta = \frac{\pi}{4}$$
27. Find the point on the curve $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$ at which the tangents are parallel to y-axis.
28. Find the equation of the tangents to the curve

    $y = x^2 - 2x + 5$, (i) which is parallel to the line $2x + y + 7 = 0$ (ii) which is perpendicular to the line $5(y - 3x) = 12$
29. Show that the tangents to the curve $y = 7x^3 + 11$ at the points $x = 2$ and $x = -2$ are parallel
30. Find the equation of normal at the point

    $$\left(am^2, am^3\right) \text{ to the curve } ay^2 = x^3$$
31. Show that $f(x) = x^2$ is neither increasing nor decreasing for all $x \in R$.

Find the intervals for which the following functions are increasing or decreasing :

32. $2x^3 - 3x^2 - 12x + 6$
33. $\dfrac{x}{4} + \dfrac{4}{x}, x \neq 0$
34. $x^4 - 2x^2$
35. $\sin x - \cos x, 0 \leq x \leq 2\pi$

Find the local maxima or minima of the following functions :

36. (a) x^3-6x^2+9x+7 (b) $2x^3-24x+107$

(c) x^3+4x^2-3x+2 (d) $x^4-62x^2+120x+9$

37. (a) $\dfrac{1}{x^2+2}$ (b) $\dfrac{x}{(x-1)(x-4)}, 1<x<4$

(c) $x\sqrt{1-x}, \quad x<1$

38. (a) $\sin x+\dfrac{1}{2}\cos 2x, \quad 0\le x\le\dfrac{\pi}{2}$ (b) $\sin 2x \quad , 0\le x\le 2\pi$

(c) $-x+2\sin x, 0\le x\le 2\pi$

39. For what value of x lying in the closed interval [0,5], the slope of the tangent to

$$x^3-6x^2+9x+4$$

is maximum. Also, find the point.

40. Find the vlaue of the greatest slope of a tangent to

$-x^3+3x^2+2x-27$ at a point of othe curve. Find also the point.

41. A container is to be made in the shape of a right circular cylinder with total surface area of 24π sq. m. Determine the dimensions of the container if the volume is to be as large as possible.

42. A hotel complex consisting of 400 two bedroom apartments has 300 of them rented and the rent is Rs. 360 per day. Management's research indicates that if the rent is reduced by x rupees then the number of apartments rented q will be $q=\dfrac{5}{4}x+300, 0\le x\le 80$. Determine the rent that results in maximum revenue. Also find the maximum revenue.

ANSWERS

CHECK YOUR PROGRESS 29.1

1.	64 cm^2/minute	2.	900 cm^3/sec	3.	12π cm^2/cm
4.	11.2π cm^2/sec	5.	75 cm^3/cm		

CHECK YOUR PROGRESS 29.2

1.	6.05	2.	2.926	3.	1.96875
4.	5.1	5.	3.92π m^3	6.	3%

CHECK YOUR PROGRESS 29.3

1. (i) $10, -\frac{1}{10}$ (ii) $-\frac{2}{5}, \frac{5}{2}$ (iii) $1, -1$

2. $p = 5, \ q = -4$ 3. $(3, 3), (-3, -3)$ 4. $(3, 2)$

CHECK YOUR PROGRESS 29.4

1.

	Tangent	**Normal**
(i)	$y + 10x = 5$	$x - 10y + 50 = 0$
(ii)	$2x - y = 1$	$x + 2y - 3 = 0$
(iii)	$24x - y = 52$	$x + 24y = 483$

2. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ 3. $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$

4. $x + 14y - 254 = 0, \ x + 14y + 86 = 0$

CHECK YOUR PROGRESS 29.6

2. $\left(\frac{1}{196}, \frac{-43}{14}\right)$

CHECK YOUR PROGRESS 29.8

1. (a) Increasing for $x > \frac{7}{2}$, Decreasing for $x < \frac{7}{2}$

   (b) Increasing for $x > \frac{5}{2}$, Decreasing for $x < \frac{5}{2}$

2. (a) Increasing for $x > 6$ or $x < -2$, Decreasing for $-2 < x < 6$

   (b) Increasing for $x > 4$ or $x < 2$, Decreasing for x in the interval]2,4[

3. (a) Increasing for $x < -2$; decreasing for $x > -2$

   (b) Increasing in the interval $-1 < x < -2$, Decreasing for $x > -1$ or $x < -2$

4. (a) Increasing always.

   (b) Increasing for $x > 2$, Decreasing in the interval $0 < x < 2$

   (c) Increasing for $x > 2$ or $x < -2$ Decreasing in the interval $-2 < x < 2$

5. (c) Increasing in the interval $\frac{3\pi}{8} \le x \le \frac{7\pi}{8}$

   Decreasing in the interval $0 \le x \le \frac{3\pi}{8}$

   Points at which the tangents are parallel to x-axis are $x = \frac{3\pi}{8}$ and $x = \frac{7\pi}{8}$

CHECK YOUR PROGRESS 29.9

1. Local minimum is – 4 at x = 4
2. Local minimum is 15 at x = 3, Local maximum is 19 at x = 1.
3. Local minimum is – 128 at x = 6, Local maximum is – 3 at x = 1.
4. Local minimum is – 1647 at x = – 6, Local minimum is – 316 at x = 5,

   Local maximum is 68 at x = 1.
5. Local minimum at x = 0 is – 4, Local maximum at x = – 2 is 0.
6. Local minimum at x = – 1, value $= -1$ Local maximum at x = 3, value $= \frac{1}{7}$

CHECK YOUR PROGRESS 29.10

1. Local minimum is – 34 at x = 2, Local maximum is 91 at x = – 3.
2. Local minimum is – 5 at x = 0, Local maximum is 251 at x = 8.
3. Local minimum – 4 at x = 0, Local maximum 0 at x = – 2.
4. Local minimum = – 28; x = 3, Local maximum = 0 ; x = 1.

   Neither maximum nor minimum at x = 0.
5. Local maximum $= \frac{3\sqrt{3}}{4}; x = \frac{\pi}{3}$
6. Local maximum $= \sqrt{2}; x = \frac{\pi}{4}$
7. Local minimum $= \frac{-\sqrt{3}}{2} + \frac{\pi}{6}; x = -\frac{\pi}{6}$, Local maximum $= \frac{\sqrt{3}}{2} - \frac{\pi}{6}; x = \frac{\pi}{6}$

CHECK YOUR PROGRESS 29.15

1. Numbers are 6,9.
2. Parts are 7.5, 7.5
5. Dimensions are : $=\dfrac{30}{\pi+4},\dfrac{30}{\pi+4}$ meters each.
6. radius $=\left(\dfrac{50}{\pi}\right)^{\frac{1}{3}}$ cm ; height $=2\left(\dfrac{50}{\pi}\right)^{\frac{1}{3}}$ cm
7. Maximum Volume $=32\pi$ cubic meters.
9. $h=\sqrt{2}r$
10. r = 2 meters, h = 4 meters.
11. Rs. 30,00

TERMINAL EXRCISE

1.	0.8 cm/sec.	2.	1.4π cm/sec.	3.	$\dfrac{45}{26}$ km/hr.
4.	$\dfrac{1}{48\pi}$ cm/sec.	5.	4 meters/minute	6.	(4, 11), $\left(-4,\dfrac{-31}{3}\right)$
7.	52.5π cm^2/sec.	8.	36π cm^3/sec.	9.	(2, 4)
10.	$\left(1,\dfrac{5}{3}\right),\left(-1,\dfrac{1}{3}\right)$	11.	66	12.	30.015
13.	5.002	14.	7.0357		
15.	(i) 20.025	(ii)	0.49		
16.	(i) 0.0608	(ii)	2.963		
17.	(i) 4.0417	(ii)	3.0093		
18.	(i) 2.001875	(ii)	1.99875		
19.	(i) 0.2495	(ii)	0.1996		
20.	77.66	21.	$0.09x^3$ m^3 or 9%		
22.	$0.12x^2$ m^2	23.	$0.03x^3$ m^3 or 3%		
24.	–34.995	25.	(i) $\dfrac{1}{6},-6$		
	(ii) $13,-\dfrac{1}{13}$	(iii)	$1,-1$		
	(iv) 0, not defined	(v)	$-6,\dfrac{1}{6}$		

26. $2\sqrt{2}\left(x+y\right)=a;\ x+y=0$ 27. $\left(3,0\right),\left(-3,0\right)$

28. (i) $2x+y-5=0$ (ii) $12x+36y=155$

30. $2x+3my-am^2\left(2+3m^2\right)=0$

32. Increasing for $x>2$ or <-1, Decreasing in the interval $-1<x<2$

33. Increasing for $x>4$ or $x<-4$, Decreasing in the interval $]-4,4[$

34. Increasing for $x>1$ or $-1<x<0$, Decreasing for $x<-1$ or $0<x<1$

35. Increasing for $0\le x\le\frac{3\pi}{4}$ or $\frac{7\pi}{4}\le x\le 2\pi$, Decreasing for $\frac{3\pi}{4}\le x\le\frac{7\pi}{4}$.

36. (a) Local maximum is 11 at $x=1$; local minimum is 7 at $x=3$.

(b) Local maximum is 139 at $x=-2$; local minimum is 75 at $x=2$.

(c) Local maximum is 20 at $x=-3$; local minimum is $\frac{40}{27}$ at $x=\frac{1}{3}$.

(d) Local maximum is 68 at $x=1$; local minimum is -316 at $x=5$ and -1647 at $x=-6$.

37. (a) Local minimum is $\frac{1}{2}$ at $x=0$.

(b) Local maximum is -1 at $x=2$.

(c) Local maximum is $\frac{2}{3\sqrt{3}}$ at $x=\frac{2}{3}$.

38. (a) Local maximum is $\frac{3}{4}$ at $x=\frac{\pi}{6}$; Local minimum is $\frac{1}{2}$ at $x=\frac{\pi}{2}$;

(b) Local maximum is 1 at $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$; Local minimum is -1 at $x=\frac{3\pi}{4}$

(c) Local maximum is $\frac{-\pi}{4}+\sqrt{3}$ at $x=\frac{\pi}{3}$; Local minimum is

$\frac{-5\pi}{3}-\sqrt{3}$ at $x=\frac{5\pi}{3}$.

39. Greatest slope is 24 at $x=5$; Coordinates of the point: (5, 24).

40. Greatest slope of a tangent is 5 at $x=1$, The point is $(1,-23)$

41. Radius of base = 2 m, Height of cylinder = 4 m.

42. Rent reduced to Rs. 300, The maximum revenue = Rs. 1,12,500.

30

INTEGRATION

In the previous lesson, you have learnt the concept of derivative of a function. You have also learnt the application of derivative in various situations.

Consider the reverse problem of finding the original function, when its derivative (in the form of a function) is given. This reverse process is given the name of integration. In this lesson, we shall study this concept and various methods and techniques of integration.

OBJECTIVES

After studying this lesson, you will be able to :

- explain integration as inverse process (anti-derivative) of differentiation;
- find the integral of simple functions like x^n, $\sin x$, $\cos x$,

  $\sec^2 x$, $\operatorname{cosec}^2 x$, $\sec x \tan x$, $\operatorname{cosec} x \cot x$, $\frac{1}{x}$, e^x etc.;
- state the following results :

  (i) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

  (ii) $\int [\pm k f(x)]\,dx = \pm k \int f(x)\,dx$
- find the integrals of algebraic, trigonometric, inverse trigonometric and exponential functions;
- find the integrals of functions by substitution method.
- evaluate integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{a^2 - x^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c},$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{(px + q)\,dx}{ax^2 + bx + c}, \int \frac{(px + q)\,dx}{\sqrt{ax^2 + bx + c}}$$

- derive and use the result

$$\int \frac{f'(x)}{f(x)} = \ell n\,|f(x)| + C$$

- state and use the method of integration by parts;

MODULE - VIII
Calculus

Notes

- evaluate integrals of the type :

$$\int \sqrt{x^2 \pm a^2}\,dx,\ \int \sqrt{a^2 - x^2}\,dx,\ \int e^{ax} \sin bx\ dx,\ \int e^{ax} \cos bx\ dx,$$

$$\int (px + q)\sqrt{ax^2 + bx + c}\ dx,\ \int \sin^{-1} x\ dx,\ \int \cos^{-1} x\ dx,$$

$$\int \sin^n x \cos^m x\ dx,\ \int \frac{dx}{a + b\sin x},\ \int \frac{dx}{a + b\cos x}$$

- derive and use the result

$$\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + c; \text{ and}$$

- integrate rational expressions using partial fractions.

EXPECTED BACKGROUND KNOWLEDGE

- Differentiation of various functions
- Basic knowledge of plane geometry
- Factorization of algebraic expression
- Knowledge of inverse trigonometric functions

30.1 INTEGRATION

Integration literally means summation. Consider, the problem of finding area of region ALMB as shown in Fig. 30.1.

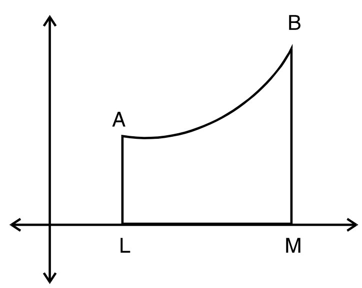


Fig. 30.1

We will try to find this area by some practical method. But that may not help every time. To solve such a problem, we take the help of integration (summation) of area. For that, we divide the figure into small rectangles (See Fig.30.2).

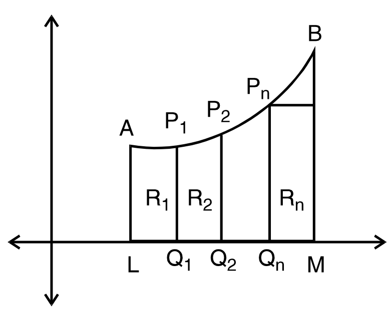


Fig. 30.2

Unless these rectangles are having their width smaller than the smallest possible, we cannot find the area.

This is the technique which Archimedes used two thousand years ago for finding areas, volumes, etc. The names of Newton (1642-1727) and Leibnitz (1646-1716) are often mentioned as the creators of present day of Calculus.

The integral calculus is the study of integration of functions. This finds extensive applications in Geometry, Mechanics, Natural sciences and other disciplines.

In this lesson, we shall learn about methods of integrating polynomial, trigonometric, exponential and logarithmic and rational functions using different techniques of integration.

30.2. INTEGRATION AS INVERSE OF DIFFERENTIATION

Consider the following examples :

(i) $\frac{d}{dx}\left(x^2\right) = 2x$ (ii) $\frac{d}{dx}(\sin x) = \cos x$ (iii) $\frac{d}{dx}\left(e^x\right) = e^x$

Let us consider the above examples in a different perspective

(i) 2x is a function obtained by differentiation of x^2.

$\Rightarrow x^2$ is called the antiderivative of $2x$

(ii) cos x is a function obtained by differentiation of sin x

$\Rightarrow \sin x$ is called the antiderivative of cos x

(iii) Similarly, e^x is called the antiderivative of e^x

Generally we express the notion of antiderivative in terms of an operation. This operation is called the operation of integration. We write

1. Integration of 2 x is x^2 2. Integration of cos x is sin x

3. Integration of e^x is e^x

The operation of integration is denoted by the symbol $\int$.

Thus

1. $\int 2x\, dx = x^2$ 2. $\int \cos x\, dx = \sin x$ 3. $\int e^x\, dx = e^x$

Remember that dx is symbol which together with symbol $\int$ denotes the operation of integration.

The function to be integrated is enclosed between $\int$ and dx.

Definition : If $\frac{d}{dx}[f(x)] = f'(x)$, then f (x) is said to be an integral of $f'(x)$ and is written as $\int f'(x)dx = f(x)$

The function f '(x) which is integrated is called the integrand.

Constant of integration

$$\text{If } \; y = x^2, \text{ then } \frac{dy}{dx} = 2x$$

$$\therefore \qquad \int 2x\,dx = x^2$$

Now consider $\frac{d}{dx}\left(x^2 + 2\right)$ or $\frac{d}{dx}\left(x^2 + c\right)$ where c is any real constant . Thus, we see that integral of 2x is not unique. The different values of $\int 2\,x\,d\,x$ differ by some constant. Therefore, $\int 2\,x\,d\,x = x^2 + C$, where c is called the constant of integration.

Thus $\int e^x\,d\,x = e^x + C$, $\int \cos x\,d\,x = \sin x + c$

In general $\int f\,'(x)\,d\,x = f\,(\,x\,) + C$. The constant c can take any value.

We observe that the derivative of an integral is equal to the integrand.

Note : $\int f(x)\,dx$, $\int f(y)\,dy$, $\int f(z)\,dz$ but not like $\int f(z)\,dx$

	Integral	**Verification**
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where n is a constant and $n \neq -1$.	$\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n$
2.	$\int \sin x\,dx = -\cos x + C$	$\because \frac{d}{dx}(-\cos x + C) = \sin x$
3.	$\int \cos x\,dx = \sin x + C$	$\because \frac{d}{dx}(\sin x + C) = \cos x$
4.	$\int \sec^2 x\,dx = \tan x + C$	$\because \frac{d}{dx}(\tan x + C) = \sec^2 x$
5.	$\int \operatorname{cosec}^2 x\,dx = -\cot x + C$	$\because \frac{d}{dx}(-\cot x + C) = \operatorname{cosec}^2 x$
6.	$\int \sec x\,\tan x\,dx = \sec x + C$	$\because \frac{d}{dx}(\sec x + C) = \sec x\,\tan x$
7.	$\int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x + C$	$\because \frac{d}{dx}(-\operatorname{cosec} x + C) = \operatorname{cosec} x\cot x$
8.	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$	$\because \frac{d}{dx}\left(\sin^{-1} x + C\right) = \frac{1}{\sqrt{1-x^2}}$

9. $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$ $\qquad \because \frac{d}{dx}\left(\tan^{-1}x + C\right) = \frac{1}{1+x^2}$

10. $\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x + C$ $\qquad \because \frac{d}{dx}\left(\sec^{-1}x + C\right) = \frac{1}{x\sqrt{x^2-1}}$

11. $\int e^x dx = e^x + C$ $\qquad \because \frac{d}{dx}\left(e^x + C\right) = e^x$

12. $\int a^x dx = \frac{a^x}{\log a} + C$ $\qquad \because \frac{d}{dx}\left(\frac{a^x}{\log a} + C\right) = a^x = \frac{1}{x}$ if $x > 0$

13. $\int \frac{1}{x}dx = \log|x| + C$ $\qquad \because \frac{d}{dx}\left(\log|x| + C\right)$

WORKING RULE

1. To find the integral of x^n, increase the index of x by 1, divide the result by new index and add constant C to it.

2. $\int \frac{1}{f(x)}dx$ will be very often written as $\int \frac{dx}{f(x)}$.

CHECK YOUR PROGRESS 30.1

1. Write any five different values of $\int x^{\frac{5}{2}}dx$

2. Write indefinite integral of the following :

   (a) x^5 (b) $\cos x$ (c) 0

3. Evaluate :

   (a) $\int x^6 dx$ (b) $\int x^{-7}dx$ (c) $\int \frac{1}{x}dx$ (d) $\int 3^x 5^{-x}dx$

   (e) $\int \sqrt[3]{x}\,dx$ (f) $\int x^{-9}\,dx$ (g) $\int \frac{1}{\sqrt{x}}dx$ (h) $\int \sqrt[9]{x^{-8}}\,dx$

4. Evaluate :

   (a) $\int \frac{\cos\theta}{\sin^2\theta}d\theta$ (b) $\int \frac{\sin\theta}{\cos^2\theta}d\theta$

(c) $\int \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\, d\theta$ (d) $\int \frac{1}{\sin^2\theta}\, d\theta$

30.4 PROPERTIES OF INTEGRALS

If a function can be expressed as a sum of two or more functions then we can write the integral of such a function as the sum of the integral of the component functions, e.g. if $f(x) = x^7 + x^3$, then

$$\int f(x)\,dx = \int [x^7 + x^3]\,dx = \int x^7 dx + \int x^3 dx = \frac{x^8}{8} + \frac{x^4}{4} + C$$

So, in general the integral of the sum of two functions is equal to the sum of their integrals.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Similarly, if the given function

$$f(x) = x^7 - x^2$$

we can write it as $\int f(x)\,dx = \int (x^7 - x^2)\,dx = \int x^7 dx - \int x^2 dx$

$$= \frac{x^8}{8} - \frac{x^3}{3} + C$$

The integral of the difference of two functions is equal to the difference of their integrals.

i.e. $$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

If we have a function f(x) as a product of a constant (k) and another function $[g(x)]$

i.e. $f(x) = kg(x)$, then we can integrate f(x) as

$$\int f(x)\,dx = \int kg(x)\,dx = k\int g(x)\,dx$$

Integral of product of a constant and a function is product of that constant and integral of the function.

i.e. $$\int kf(x)\,dx = k\int f(x)\,dx$$

Example 30.1 Evaluate :

(ii) $\int 4^x dx$ (ii) $\int (2^x)(3^{-x})\,dx$

Solution :(i) $\int 4^x dx = \frac{4^x}{\log 4} + C$

(ii) $$\int (2^x)(3^{-x})\,dx = \int \frac{2^x}{3^x}\,dx = \int \left(\frac{2}{3}\right)^x dx = \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$$

Remember in (ii) it would not be correct to say that

$$\int 2^x 3^{-x}\,dx = \int 2^x\,dx \int 3^{-x}\,dx$$

Because $$\int 2^x\,dx \int 3^{-x}\,dx = \frac{2^x}{\log 2}\left(\frac{3^{-x}}{\log 3}\right) + C \neq \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$$

Therefore, **integral of a product of two functions is not always equal to the product of the integrals.** We shall deal with the integral of a product in a subsequent lesson.

Example 30.2 Evaluate :

(i) $\int \frac{dx}{\cos^n x}$, when n = 0 and n = 2 (ii) $\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\,d\theta$

Solution :

(i) When n = 0, $$\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^o x}$$

$$= \int \frac{dx}{1} = \int dx$$

Now $\int dx$ can be written as $\int x^o dx$.

$\therefore$ $$\int dx = \int x^o dx = \frac{x^{o+1}}{0+1} + C = x + C$$

When n = 2,

$$\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^2 x}$$

$$= \int \sec^2 x\,dx$$

$$= \tan x = C$$

(ii) $$\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{-1}{\sin^2\theta}\,d\theta = -\int \operatorname{cosec}^2\theta\,d\theta$$

$$= \cot\theta + C$$

Example 30.3 Evaluate :

(i) $\int (\sin x + \cos x)\,dx$ (ii) $\int \frac{x^2+1}{x^3}\,dx$

(iii) $\int \frac{1-x}{\sqrt{x}}\,dx$ (iv) $\int \left(\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx$

Solution : (i) $\int (\sin x + \cos x)\,dx = \int \sin x\,dx + \int \cos x\,dx = -\cos x + \sin x + C$

(ii) $$\int \frac{x^2+1}{x^3}\,dx = \int \left(\frac{x^2}{x^3} + \frac{1}{x^3} \right) dx = \int \frac{1}{x}\,dx + \int \frac{1}{x^3}\,dx$$

$$= \log|x| + \frac{x^{-3+1}}{-3+1} + C = \log|x| - \frac{1}{2x^2} + C$$

(iii) $$\int \frac{1-x}{\sqrt{x}}\,dx = \int \left(\frac{1}{\sqrt{x}} - \frac{x}{\sqrt{x}} \right) dx = \int \left(x^{-\frac{1}{2}} - x^{\frac{1}{2}} \right) dx$$

$$= 2\sqrt{x} - \frac{2}{3}x^{3/2} + C$$

(iv) $$\int \left(\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx = \int \frac{dx}{1+x^2} - \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \tan^{-1} x - \sin^{-1} x + C$$

Example 30.4 Evaluate :

(i) $\int \sqrt{1-\sin 2\theta}\;d\theta$ (ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2-1}} \right) dx$

(iii) $\int (\tan x + \cot x)^2\,dx$ (iv) $\int \left(\frac{x^6-1}{x^2-1} \right) dx$

Solution : (i) $\sqrt{1-\sin 2\theta} = \sqrt{\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}$

$$\left[\because \sin^2\theta + \cos^2\theta = 1 \right]$$

$$= \sqrt{(\cos\theta - \sin\theta)^2} = \pm(\cos\theta - \sin\theta)$$

(sign is selected depending upon the value of θ)

(a) If $\sqrt{1-\sin 2\theta} = \cos\theta - \sin\theta$

then $\int \sqrt{1-\sin 2\theta}\;d\theta = \int (\cos\theta - \sin\theta)\,d\theta$

$$= \int \cos\theta \, d\theta - \int \sin\theta \, d\theta = \sin\theta + \cos\theta + C$$

(b) If $\int \sqrt{1 - \sin 2\theta} \, d\theta = \int (-\cos\theta + \sin\theta) \, d\theta = -\int \cos\theta \, d\theta + \int \sin\theta \, d\theta$

$$= -\sin\theta - \cos\theta + C$$

(ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2 - 1}} \right) dx = \int 4e^x dx - \int \frac{3}{x\sqrt{x^2 - 1}} dx$

$$= 4\int e^x dx - 3\int \frac{dx}{x\sqrt{x^2 - 1}} = 4e^x - 3\sec^{-1} x + C$$

(iii) $\int (\tan x + \cot x)^2 dx = \int \left(\tan^2 x + \cot^2 x + 2 \tan x \cot x \right) dx$

$$= \int \left(\tan^2 x + \cot^2 x + 2 \right) dx$$

$$= \int \left(\tan^2 x + 1 + \cot^2 x + 1 \right) dx$$

$$= \int \left(\sec^2 x + \operatorname{cosec}^2 x \right) dx$$

$$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx$$

$$= \tan x - \cot x + C$$

(iv) $\int \left(\frac{x^6 - 1}{x^2 + 1} \right) dx = \int \left(x^4 - x^2 + 1 - \frac{2}{x^2 + 1} \right) dx$ (dividing $x^6 - 1$ by $x^2 + 1$)

$$= \int x^4 dx - \int x^2 dx + \int dx - 2\int \frac{dx}{x^2 + 1}$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1} x + C$$

Example 30.5 Evaluate :

(i) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^3 dx$ (ii) $\int \left(\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}} \right) dx$

Solution :

(i) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^3 dx = \int \left(x^{3/2} + 3x \frac{1}{\sqrt{x}} + 3\sqrt{x} \frac{1}{x} + \frac{1}{x^{3/2}} \right) dx$

$$= \int x^{3/2} dx + 3\int \sqrt{x} \, dx + 3\int \frac{1}{\sqrt{x}} dx + \int \frac{dx}{x^{3/2}}$$

$$= \frac{x^{5/2}}{\frac{5}{2}} + 3\frac{x^{3/2}}{\frac{3}{2}} + 3\frac{x^{1/2}}{\frac{1}{2}} - \frac{2}{\sqrt{x}} + C$$

$$= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + C$$

(ii) $$\int \left(\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}} \right) dx = \int \frac{4e^{5x}}{e^{3x}} dx - \int \frac{9e^{4x}}{e^{3x}} dx - \int \frac{3dx}{e^{3x}}$$

$$= 4\int e^{2x}\, dx - 9\int e^{x}\, dx - 3\int e^{-3x}\, dx$$

$$= 2e^{2x} - 9e^{x} + e^{-3x} + C$$

CHECK YOUR PROGRESS 30.2

1. Evaluate :

(a) $\int \left(x + \frac{1}{2} \right) dx$ (b) $\int \frac{-x^2}{1 + x^2} dx$

(c) $\int \left(10x^9 - \sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$ (d) $\int \left(\frac{5 + 3x - 6x^2 - 7x^4 - 8x^6}{x^6} \right) dx$

(e) $\int \frac{x^4}{1 + x^2} dx$ (f) $\int \left(\sqrt{x} + \frac{2}{\sqrt{x}} \right)^2 dx$

2. Evaluate :

(a) $\int \frac{dx}{1 + \cos 2x}$ (b) $\int \tan^2 x dx$ (c) $\int \frac{2\cos x}{\sin^2 x} dx$

(d) $\int \frac{dx}{1 - \cos 2x}$ (e) $\int \frac{\sin x}{\cos^2 x} dx$ (f) $\int (\operatorname{cosec} x - \cot x) \operatorname{cosec} x\, dx$

3. Evaluate :

(a) $\int \sqrt{1 + \cos 2x}\, dx$ (b) $\int \sqrt{1 - \cos 2x}\, dx$ (c) $\int \frac{1}{1 - \cos 2x} dx$

4. Evaluate :

(a) $\int \sqrt{x + 2}\, dx$

30.5 TECHNIQUES OF INTEGRATION

11.5.1 Integration By Substitution

This method consists of expressing $\int f(x) dx$ in terms of another variable so that the resultant function can be integrated using one of the standard results discussed in the previous lesson. First, we will consider the functions of the type $f(ax + b)$, $a \neq 0$ where f (x) is a standard function.

Example 30.6 Evaluate :

(i) $\int \sin(ax+b)\,dx$

Solution : (i) $\int \sin(ax+b)\,dx$

Put $\quad ax+b=t.$

Then $\quad a=\dfrac{dt}{dx} \quad$ or $\quad dx=\dfrac{dt}{a}$

$\therefore \quad \int \sin(ax+b)\,dx=\int \sin t\,\dfrac{dt}{a}$ (Here the integration factor will be replaced by dt.)

$$=\frac{1}{a}\int \sin t\,dt=\frac{1}{a}(-\cos t)+C=-\frac{\cos(ax+b)}{a}+C$$

Example 30.7 Evaluate :

(i) $\int (ax+b)^n\,dx$, where $n\neq -1$ $\qquad$ (ii) $\int \dfrac{1}{(ax+b)}\,dx$

Solution : (i) $\int (ax+b)^n\,dx$, where $n\neq -1$

Put $\quad ax+b=t \quad \Rightarrow \quad a=\dfrac{dt}{dx}$ or $dx=\dfrac{dt}{a}$

$$\therefore \quad \int (ax+b)^n\,dx=\frac{1}{a}\int t^n dt=\frac{1}{a}\cdot\frac{t^{n+1}}{(n+1)}+C$$

$$=\frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1}+C \qquad \text{where } n\neq -1$$

(ii) $\int \dfrac{1}{(ax+b)}\,dx$

Put $\quad ax+b=t \quad \Rightarrow \quad dx=\dfrac{1}{a}dt$

$$\therefore \quad \int \frac{1}{(ax+b)}\,dx=\int \frac{1}{a}\cdot\frac{dt}{t}=\frac{1}{a}\log|t|+C$$

$$=\frac{1}{a}\log|ax+b|+C$$

Example 30.8 Evaluate :

(i) $\int e^{5x+7} dx$

Solution : (i) $\int e^{5x+7} dx$

Put $5x + 7 = t \Rightarrow dx = \dfrac{dt}{5}$

$$\therefore \quad \int e^{5x+7} dx = \frac{1}{5}\int e^t dt = \frac{1}{5}e^t + C$$

$$= \frac{1}{5}e^{5x+7} + C$$

Likewise $\int e^{ax+b} dx = \dfrac{1}{a}e^{ax+b} + C$

Similarly, using the substitution $ax + b = t$, the integrals of the following functions will be :

$$\int (ax+b)^n\, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{(ax+b)} dx = \frac{1}{a}\log|ax+b| + C$$

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \mathrm{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + C$$

$$\int \sec(ax+b)\tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b) + C$$

$$\int \mathrm{cosec}(ax+b)\cot(ax+b)\, dx = -\frac{1}{a}\mathrm{cosec}(ax+b) + C$$

Example 30.9 Evaluate :

(i) $\int \sin^2 x\, dx$ (ii) $\int \sin^3 x\, dx$ (iii) $\int \cos^3 x\, dx$ (iv) $\int \sin 3x \sin 2x\, dx$

Solution : We use trigonometrical identities and express the functions in terms of sines and cosines of multiples of x

(i) $\int \sin^2 x\ dx = \int \frac{1-\cos 2x}{2}\ dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$

$= \frac{1}{2}\int (1-\cos 2x)\ dx \ = \frac{1}{2}\int 1dx - \frac{1}{2}\int \cos 2x\ dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(ii) $\int \sin^3 x\ dx = \int \frac{3\sin x - \sin 3x}{4} dx \quad \left[\because \ \sin 3x = 3\sin x - 4\sin^3 x\right]$

$= \frac{1}{4}\int (3\sin x - \sin 3x)\ dx \ = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$

(iii) $\int \cos^3 x\ dx = \int \frac{\cos 3x + 3\cos x}{4}\ dx \ \left[\because \cos 3x = 4\cos^3 x - 3\cos x\right]$

$= \frac{1}{4}\int (\cos 3x + 3\cos x)\ dx \ = \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right] + C$

(iv) $\int \sin 3x\ \sin 2x\ dx = \frac{1}{2}\int 2\sin 3x\ \sin 2x\ dx$

$[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$

$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx \ = \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right] + C$

CHECK YOUR PROGRESS 30.3

1. Evaluate :

(a) $\int \sin(4-5x)\, dx$ (b) $\int \sec^2(2+3x)\, dx$

(c) $\int \sec\left(x + \frac{\pi}{4}\right) dx$ (d) $\int \cos(4x+5)\, dx$

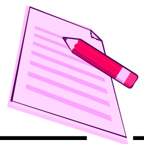
(e) $\int \sec(3x+5)\tan(3x+5)\, dx$

(f) $\int \operatorname{cosec}(2+5x)\cot(2+5x)\, dx$

2. Evaluate :

(a) $\int \frac{dx}{(3-4x)^4}$ (b) $\int (x+1)^4\, dx$ (c) $\int (4-7x)^{10}\, dx$

(d) $\int (4x-5)^3\, dx$ (e) $\int \frac{1}{3x-5} dx$ (f) $\int \frac{1}{\sqrt{5-9x}} dx$



(g) $\int (2x+1)^2\, dx$ (h) $\int \frac{1}{x+1}\, dx$

3. Evaluate :

(a) $\int e^{2x+1} dx$ (b) $\int e^{3-8x} dx$ (c) $\int \frac{1}{e^{(7+4x)}}\, dx$

4. Evaluate :

(a) $\int \cos^2 x\, dx$ (b) $\int \sin^3 x \cos^3 x\, dx$

(c) $\int \sin 4x \cos 3x\, dx$ (d) $\int \cos 4x \cos 2x\, dx$

30.5.2 Integration of Function of The Type $\frac{f'(x)}{f(x)}$

To evaluate $\int \frac{f'(x)}{f(x)}\, dx$, we put f (x) = t. Then f ' (x) dx = dt.

$$\therefore \quad \int \frac{f'(x)}{f(x)}\, dx = \int \frac{dt}{t} = \log|t| + C = \log|f(x)| + C$$

Integral of a function, whose numerator is derivative of the denominator, is equal to the logarithm of the denominator.

Example 30.10 Evaluate :

(i) $\int \frac{2x}{x^2+1}\, dx$ (ii) $\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)}$

Solution :

(i) Now 2x is the derivative of x^2+1.

$\therefore$ By applying the above result, we have

$$\int \frac{2x}{x^2+1}\, dx = \log\left|x^2+1\right| + C$$

(ii) $\frac{1}{2\sqrt{x}}$ is the derivative of $3+\sqrt{x}$

$$\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)} = \log\left|3+\sqrt{x}\right| + C$$

Example 30.11 Evaluate :

(i) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$ (ii) $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$

Solution :

(i) $e^x + e^{-x}$ is the derivative of $e^x - e^{-x}$

$$\therefore \quad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \log\left|e^x - e^{-x}\right| + C$$

Alternatively,

For $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$,

Put $e^x - e^{-x} = t.$

Then $\left(e^x + e^{-x}\right) dx = dt$

$$\therefore \quad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int \frac{dt}{t} = \log|t| + C = \log\left|e^x - e^{-x}\right| + C$$

(ii) $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$

Here $e^{2x} - 1$ is not the derivative of $e^{2x} + 1$. But if we multiply the numerator and denominator by e^{-x}, the given function will reduce to

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \log\left|e^x + e^{-x}\right| + C$$

$$\therefore \quad \int \frac{e^{2x} - 1}{e^{2x} + 1} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} = \log\left|e^x + e^{-x}\right| + C$$

$$\left[\because \left(e^x - e^{-x}\right) \text{ is the derivative of } \left(e^x + e^{-x}\right)\right]$$

CHECK YOUR PROGRESS 30.4

1. Evaluate :

(a) $\int \frac{x}{3x^2 - 2} dx$ (b) $\int \frac{2x + 1}{x^2 + x + 1} dx$ (c) $\int \frac{2x + 9}{x^2 + 9x + 30} dx$

(d) $\int \frac{x^2+1}{x^3+3x+3}\,dx$ (e) $\int \frac{2x+1}{x^2+x-5}\,dx$ (f) $\int \frac{dx}{\sqrt{x}\left(5+\sqrt{x}\right)}$

(g) $\int \frac{dx}{x\,(8+\log x)}$

2. Evaluate :

(a) $\int \frac{e^x}{2+be^x}\,dx$ (b) $\int \frac{dx}{e^x-e^{-x}}$

30.5.3 INTEGRATION BY SUBSTITUTION

(i) $\int \tan x\,dx$ (ii) $\int \sec x\,dx$

Solution :

(i) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx$

$= -\log\left|\cos x\right| + C$ ($\because -\sin x$ is derivative of $\cos x$)

$= \log\left|\frac{1}{\cos x}\right| + C$ or $= \log\left|\sec x\right| + C$

$\therefore$ $\int \tan x\,dx = \log\left|\sec x\right| + C$

Alternatively,

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\int \frac{-\sin x\,dx}{\cos x}$$

Put $\cos x = t.$

Then $-\sin x\,dx = dt$

$\therefore$ $\int \tan x\,dx = -\int \frac{dt}{t} = -\log\left|t\right| + C = -\log\left|\cos x\right| + C$

$$= \log\left|\frac{1}{\cos x}\right| + C = \log\left|\sec x\right| + C$$

(ii) $\int \sec x\,dx$

$\sec x$ can not be integrated as such because $\sec x$ by itself is not derivative of any function. But this is not the case with $\sec^2 x$ and $\sec x \tan x$. Now $\int \sec x\,dx$ can be written as

$$\int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$$

$$= \int \frac{\left(\sec^2 x + \sec x \tan x\right)}{\sec x + \tan x} dx$$

Put $\quad \sec x + \tan x = t.$

Then $\quad \left(\sec x \tan x + \sec^2 x\right) dx = dt$

$$\therefore \quad \int \sec x \, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$$

Example 30.13 Evaluate $\int \frac{1}{a^2 - x^2} dx$

Solution : Put $x = a \sin\theta \quad \Rightarrow \quad dx = a\cos\theta \, d\theta$

$$\therefore \quad \int \frac{1}{a^2 - x^2} dx = \int \frac{a\cos\theta}{a^2 - a^2\sin^2\theta} d\theta$$

$$= \frac{1}{a}\int \frac{\cos\theta}{1 - \sin^2\theta} d\theta = \frac{1}{a}\int \frac{1}{\cos\theta} d\theta = \frac{1}{a}\int \sec\theta \, d\theta$$

$$= \frac{1}{a}\log|\sec\theta + \tan\theta| + C = \frac{1}{a}\log\left|\frac{1+\sin\theta}{\cos\theta}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1 + \frac{x}{a}}{\sqrt{1 - \frac{x^2}{a^2}}}\right| + C = \frac{1}{a}\log\left|\frac{a+x}{\sqrt{a^2 - x^2}}\right| + C = \frac{1}{a}\log\left|\frac{\sqrt{a+x}}{\sqrt{a-x}}\right| + C$$

$$= \frac{1}{a}\log\left|\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\right| + C = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

Example 30.14 Evaluate : $\int \frac{1}{x^2 - a^2} dx$

Solution : Put $x = a\sec\theta \quad \Rightarrow \quad dx = a\sec\theta\tan\theta \, d\theta$

$$\therefore \quad \int \frac{1}{x^2 - a^2} dx = \int \frac{a\sec\theta\tan\theta \, d\theta}{a^2\sec^2\theta - a^2}$$

$$= \frac{1}{a}\int \frac{\sec\theta\tan\theta}{\tan^2\theta} d\theta \qquad \left(\tan^2\theta = \sec^2\theta - 1\right)$$

$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}\, d\theta = \frac{1}{a}\int \frac{1}{\sin\theta}\, d\theta = \frac{1}{a}\int \operatorname{cosec}\theta \, d\theta$$

$$= \frac{1}{a}\log\left|\operatorname{cosec}\theta - \cot\theta\right| + C = \frac{1}{a}\log\left|\frac{1-\cos\theta}{\sin\theta}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1-\frac{a}{x}}{\sqrt{1-\frac{a^2}{x^2}}}\right| + C = \frac{1}{a}\log\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{\sqrt{x-a}}{\sqrt{x+a}}\right| + C = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

Example 30.15 $\int \frac{1}{a^2+x^2}\, dx$

Solution : Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

$$\therefore \quad \int \frac{1}{a^2+x^2}\, dx = \int \frac{a\sec^2\theta}{a^2\left(1+\tan^2\theta\right)}\, d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C \qquad \left(\frac{x}{a} = \tan\theta \Rightarrow \tan^{-1}\frac{x}{a} = \theta\right)$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Example 30.16 $\int \frac{1}{\sqrt{a^2-x^2}}\, dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

$$\therefore \quad \int \frac{1}{\sqrt{a^2-x^2}}\, dx = \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}\, d\theta$$

$$= \int \frac{a\cos\theta}{a\cos\theta}\, d\theta = \int d\theta = \theta + C$$

$$= \sin^{-1}\frac{x}{a} + C$$

Example 30.17 $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$

Solution : Let $x = a \sec\theta \quad \Rightarrow \quad dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \qquad \int \frac{1}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\,\tan\theta}{a\sqrt{\sec^2\theta - 1}}\, d\theta$$

$$= \int \sec\theta d\theta = \log\left|\sec\theta + \tan\theta\right| + C$$

$$= \log\left|\frac{x}{a} + \frac{1}{a}\sqrt{x^2 - a^2}\right| + C$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Example 30.18 $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx$

Solution : Put $x = a\tan\theta \quad \Rightarrow \quad dx = a\sec^2\theta\, d\theta$

$$= \int \sec\theta\, d\theta$$

$$= \log\left|\sec\theta + \tan\theta\right| + C \; = \log\left|\frac{1}{a}\sqrt{a^2 + x^2} + \frac{x}{a}\right| + C$$

$$= \log\left|\sqrt{a^2 + x^2} + x\right| + C$$

Example 30.19 $\int \frac{x^2 + 1}{x^4 + 1}\, dx$

Solution : Since x^2 is not the derivative of $x^4 + 1$, therefore, we write the given integral as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx$$

Let $x - \frac{1}{x} = t$. Then $\quad \therefore \quad \left(1 + \frac{1}{x^2}\right) dx = dt$

Also $x^2 - 2 + \frac{1}{x^2} = t^2 \quad \Rightarrow \quad x^2 + \frac{1}{x^2} = t^2 + 2$

$$\therefore \quad \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{dt}{t^2+2} = \int \frac{dt}{(t)^2+\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C$$

Example 30.20 $\int \frac{x^2-1}{x^4+1}\,dx$

Solution : $\int \frac{x^2-1}{x^4+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$

Put $x+\frac{1}{x} = t.$ Then $\left(1-\frac{1}{x^2}\right)dx = dt$

Also $x^2+2+\frac{1}{x^2} = t^2 \quad \Rightarrow \quad x^2+\frac{1}{x^2} = t^2-2$

$$\therefore \quad \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{dt}{t^2-2} = \int \frac{dt}{(t)^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

Example 30.21 $\int \frac{x^2}{x^4+1}\,dx$

Solution : In order to solve it, we will reduce the given integral to the integrals given in Examples 11.19 and 11.20.

i.e., $\int \frac{x^2}{x^4+1}\,dx = \frac{1}{2}\int \left[\frac{x^2+1}{x^4+1}+\frac{x^2-1}{x^4+1}\right]dx$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx + \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right] + C$$

Example 30.22 $\int \frac{1}{x^4+1}dx$

Solution : We can reduce the given integral to the following form

$$\frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right] + C$$

Example 30.23 (a) $\int \frac{1}{x^2-x+1}dx$ (b) $\int \frac{x^2-1}{x^4+x^2+1}dx$

Solution : (a) $\int \frac{1}{x^2-x+1}dx = \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}dx$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

(b) $$\int \frac{x^2-1}{x^4+x^2+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

Put $x+\frac{1}{x}=t. \quad \Rightarrow \quad \left(1-\frac{1}{x^2}\right)dx = dt$

Also $x^2+2+\frac{1}{x^2}=t^2 \quad \Rightarrow \quad x^2+1+\frac{1}{x^2}=t^2-1$

$$\therefore \quad \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx = \int \frac{dt}{t^2-1} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$$

Example 30.24 $\int \sqrt{\tan x}\,dx$

Solution : Let $\tan x = t^2 \quad \Rightarrow \quad \sec^2 x\,dx = 2t\,dt$

$$\Rightarrow \quad dx = \frac{2t}{\sec^2 x}\,dt = \frac{2t}{1+t^4}\,dt$$

$$\therefore \quad \int \sqrt{\tan x}\,dx = \int t\left(\frac{2t}{1+t^4}\right)dt = \int \frac{2t^2}{1+t^4}\,dt$$

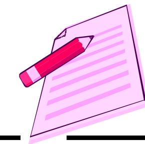
$$= \int \left(\frac{t^2+1}{t^4+1}+\frac{t^2-1}{t^4+1}\right)dt = \int \frac{t^2+1}{t^4+1}\,dt + \int \frac{t^2-1}{t^4+1}\,dt$$

Example 30.25 $\int \sqrt{\cot x}\,dx$

Solution : Let $\cot x = t^2 \quad \Rightarrow \quad -\mathrm{cosec}^2 x\,dx = 2t\,dt$

$$\Rightarrow \quad dx = \frac{-2t}{\mathrm{cosec}^2 x}\,dt = -\frac{2t}{t^4+1}\,dt$$

$$\therefore \quad \int \sqrt{\cot x}\,dx = -\int t\left(\frac{2t}{t^4+1}\right)dt$$



Notes

$$= -\int \frac{2t^2}{t^4+1}\,dt = -\int \left(\frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1} \right) dt$$

Proceed according to Examples 11.19 and 11.20 solved before.

Example 30.26 $\int \left(\sqrt{\tan x} + \sqrt{\cot x} \right) dx$

Let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$

Also $1 - 2\sin x \cos x = t^2 \Rightarrow \quad 1 - t^2 = 2\sin x \cos x$

$$\Rightarrow \quad \frac{1-t^2}{2} = \sin x \cos x$$

$$\therefore \quad \int \frac{\sin x - \cos x}{\sqrt{\cos x \sin x}}\,dx = \int \frac{dt}{\sqrt{\frac{1-t^2}{2}}} = \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\sin^{-1}[\sin x - \cos x] + C$$

(Using the result of Example 26.25)

Example 30.27 Evaluate :

(a) $\int \frac{dx}{\sqrt{8+3x-x^2}}$ (b) $\int \frac{dx}{x(1-2x)}$

Solution :

(a) $$\int \frac{dx}{\sqrt{8+3x-x^2}} = \int \frac{dx}{\sqrt{8-(x^2-3x)}}$$

$$= \int \frac{dx}{\sqrt{8-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}}$$

$$= \sin^{-1}\left[\frac{\left(x-\frac{3}{2}\right)}{\frac{\sqrt{41}}{2}} \right] + C$$

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}} \right) + C$$

(b) $$\int \frac{dx}{\sqrt{x(1-2x)}} = \int \frac{dx}{\sqrt{x-2x^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{x}{2} - x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{1}{16} - \left[x^2 - \frac{x}{2} + \frac{1}{16}\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} = \frac{1}{\sqrt{2}} \sin^{-1}\left\{\frac{\left(x - \frac{1}{4}\right)}{\left(\frac{1}{4}\right)}\right\} + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}(4x-1) + C$$

CHECK YOUR PROGRESS 30.5

1. Evaluate :

(a) $\int \frac{x^2}{x^2-9} dx$ (b) $\int \frac{e^x}{e^{2x}+1} dx$ (c) $\int \frac{x}{1+x^4} dx$

(d) $\int \frac{dx}{\sqrt{16-9x^2}}$ (e) $\int \frac{dx}{1+3\sin^2 x}$ (f) $\int \frac{dx}{\sqrt{3-2x-x^2}}$

(g) $\int \frac{dx}{3x^2+6x+21}$ (h) $\int \frac{dx}{\sqrt{5-4x-x^2}}$ (i) $\int \frac{dx}{x\sqrt{3x^2-12}}$

(j) $\int \frac{d\theta}{\sin^4\theta + \cos^4\theta}$ (k) $\int \frac{e^x dx}{\sqrt{1+e^{2x}}}$ (l) $\int \sqrt{\frac{1+x}{1-x}}\, dx$

(m) $\int \frac{dx}{\sqrt{2ax-x^2}}$ (n) $\int \frac{3x^2}{\sqrt{9-16x^6}} dx$ (o) $\int \frac{(x+1)}{\sqrt{x^2+1}} dx$

(p) $\int \frac{dx}{\sqrt{9+4x^2}}$ (q) $\int \frac{\sin\theta}{\sqrt{4\cos^2\theta - 1}} d\theta$ (r) $\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 4}} dx$

(s) $\int \frac{1}{(x+2)^2+1} dx$ (t) $\int \frac{1}{\sqrt{16x^2+25}} dx$

30.6 INTEGRATION BY PARTS

In differentiation you have learnt that

$$\frac{d}{dx}(fg) = f\frac{d}{dx}(g) + g\frac{d}{dx}(f)$$

or $$f\frac{d}{dx}(g)=\frac{d}{dx}(fg)-g\frac{d}{dx}(f) \qquad (1)$$

Also you know that $\int \frac{d}{dx}(fg)\,dx = fg$

Integrating (1). we have

$$\int f\frac{d}{dx}(g)\,dx=\int \frac{d}{dx}(fg)\,dx-\int g\frac{d}{dx}(f)\,dx$$

$$= fg-\int g\frac{d}{dx}(f)\,dx$$

if we take $$f=u(x):\frac{d}{dx}(g)=v(x).$$

(2) become $\int u(x)v(x)\,dx$

$$=u(x)\cdot\int v(x)\,dx-\int\left[\frac{d}{dx}(u(x))\int v(x)\,dx\right]dx$$

=I function $\times$ integral of II function $-\int$ [differential coefficient of function $\times$ integralof II function]dx

A B

Here the important factor is the choice of I and II function in the product of two functions because either can be I or II function. For that the indicator will be part 'B' of the result above.

The first function is to be chosen such that it reduces to a next lower term or to a constant term after subseqent differentiations.

In questions of integration like

$$x\sin x, x\cos^2 x, x^2e^x$$

(i) algebraic function should be taken as the first function

(ii) If there is no algebraic function then look for a function which simplifies the product in 'B' as above; the choice can be in order of preference like choosin first function

(i) an inverse function (ii) a logarithmic function

(iii) a trigonometric function (iv) an exponential function.

The following examples will give a practice to the concept of choosing first function.

		I function	II function
1.	$\int x\cos x\,dx$	x (being algebraic)	cos x

2.	$\int x^2 e^x dx$	x^2 (being algebraic)	e^x
3.	$\int x^2 \log dx$	log x	x^2
4.	$\int \frac{\log x}{(1+x^2)} dx$	log x	$\frac{1}{(1+x)^2}$
5.	$\int x \sin^{-1} x\, dx$	$\sin^{-1} x$	x
6.	$\int \log x\, dx$	log x	1 (In single function of logarithm and inverse trigonometric we take unity as II function)
7.	$\sin^{-1} x\, dx$	$\sin^{-1} x$	1

Example 30.28 Evaluate :

$$\int x^2 \sin x\, dx$$

Solution: Taking algebraic function x^2 as function and sin x as II function, we herv.

$$\int_{I}^{x^2} \sin x\, dx = x^2 \int \sin x - \int \left[\frac{d}{dx}(x^2) \int \sin x\, dx \right] dx$$

$$= -x^2 \cos x - 2\int x(-\cos x)\, dx$$

$$= -x^2 \cos x + 2\int x \cos x\, dx \qquad (1)$$

again $\int x \cos x\, dx = x \sin x + \cos x + c \qquad (2)$

Substituting (2) in (1), we have

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2[x \sin x + \cos x] + C$$

$$= -x^2 \cos x + 2x \sin x + \cos x + C$$

Example 30.29 Evaluate :

$$\int x^2 \log x\, dx$$

Solution: In order of preference log x is to be taken as I function.

$$\therefore \quad \int \underset{\text{I}}{\log x}\, \underset{\text{II}}{x^2}\, dx = \frac{x^3}{3}\log x - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\log x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\log x - \frac{1}{-3}\left(\frac{x^3}{3}\right) + C$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C$$

Example 30.30 Evaluate :

$$\int \sin^{-1} x\, dx$$

Solution: $\int \sin^{-1} x\, dx = \int \sin^{-1} x \cdot 1 \cdot dx$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $\quad 1 - x^2 = t \quad \Rightarrow \quad -2x\,dx = dt \quad \Rightarrow \quad x\,dx = \frac{-1}{2}dt$

$$\therefore \quad \int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} + C = -\sqrt{1-x^2} + C$$

$$\int \sin^{-1} x\, dx = x\sin^{-1} x + \sqrt{1-x^2} + C$$

CHECK YOUR PROGRESS 30.6

Evaluate:

1. (a) $\int x\sin x\,dx$ (b) $\int (1+x^2)\cos 2x\,dx$ (c) $\int x\sin 2x\,dx$

2. (a) $\int x\tan^2 x\,dx$ (b) $\int x^2 \sin^2 x\,dx$

3. (a) $\int x^3 \log 2x\,dx$ (b) $(1-x^2)\log x\,dx$ (c) $\int (\log x)^2\,dx$

4. (a) $\int \frac{\log x}{x^n}\,dx$ (b) $\int \frac{\log(\log x)}{x}\,dx$

5. (a) $\int x^2 e^{3x} dx$ (b) $\int x\, e^{3x} dx$

6. (a) $\int x(\log x)^2\, dx$

7. (a) $\int \sec^{-1} x\, dx$ (b) $\int x \cot^{-1} x\, dx$

30.7 INTEGRAL OF THE FORM

$$\int e^x \left[f(x) + f'(x) \right] dx$$

where f' (x) is the differentiation of f (x). In such type of integration while integrating by parts the solution will be $e^x \left(f(x) \right) + C.$

For example, consider

$$\int e^x \left[\tan x + \log \sec x \right] dx$$

Let $\int(x) = \log \sec x,$ then $f'(x) = \dfrac{\sec x \tan x}{\sec x} = \tan x$

So (1) can be rewritten as

$$\int e^x \left[f'(x) + f(x) \right] dx = e^x \left(f(x) \right) + C - e^x \log \sec x + C$$

Alternatively, you can evaluate it as under:

$$\int e^x \left[\tan x + \log \sec x \right] dx = \int \underset{\mathrm{I}}{e^x} \underset{\mathrm{II}}{\tan x}\, dx + \int e^x \log \sec x\, dx$$

$$= e^x \log \sec x - \int e^x \log \sec x\, dx + \int e^x \log \sec x\, dx$$

$$= e^x \log \sec x + C$$

Example 30.31 Evaluate the following:

(a) $\int e^x \left(\dfrac{1}{x} - \dfrac{1}{x^2} \right) dx$ (b) $\int e^x \left(\dfrac{1 + x \log x}{x} \right) dx$

(c) $\int \dfrac{x e^x}{(x+1)^2} dx$ (d) $\int e^x \left[\dfrac{1 + \sin x}{1 + \cos x} \right] dx$

Solution:

(a) $$\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\int e^x\left[\frac{1}{x}+\frac{d}{dx}\left(\frac{1}{x}\right)\right]dx=e^x\left(\frac{1}{x}\right)$$

(b) $$\int e^x\left(\frac{1+x\log x}{x}\right)dx=\int e^x\left(\frac{1}{x}+\log x\right)dx$$

$$=\int e^x\left(\log x+\frac{d}{dx}(\log x)\right)dx \quad =e^x\log x+C$$

(c) $$\int\frac{xe^x}{(x+1)^2}dx=\int\frac{x+1-1}{(x+1)^2}e^x\,dx \quad =\int e^x\left(\frac{1}{x+1}-\frac{1}{(x+1)^2}\right)dx$$

$$=\int e^x\left(\frac{1}{x+1}-\frac{d}{dx}\left(\frac{1}{(x+1)}\right)\right)dx$$

$$=e^x\left(\frac{1}{x+1}\right)+C$$

(d) $$=\int e^x\left[\frac{1+\sin x}{1+\cos x}\right]dx=\int e^x\left[\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right]dx$$

$$=\int e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right]dx$$

$$=\int e^x\left[\tan\frac{x}{2}+\frac{d}{x}\left(\tan\frac{x}{2}\right)\right]dx$$

$$=e^x\tan\frac{x}{2}+C$$

Example 30.32 Evaluate the following:

(a) $\int\sec^3 x\,dx$ (b) $\int e^x\sin x\,dx$

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Notes

Solution:

(a) $$\int \sec^3 x\,dx$$

Let $$I = \int \sec x \cdot \sec^2 x\,dx$$

$$= \sec x \cdot \tan x - \int \sec x \tan x \cdot \tan x\,dx$$

$\therefore$ $$I = \sec x \tan x - \int \left(\sec^3 x - \sec x\right)dx \qquad \left(\because \tan^2 x = \sec^2 x - 1\right)$$

or $$I = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

or $$2I = \sec x \tan x + \int \sec x\,dx$$

or $$I = \sec x \tan x + \log\left|\sec x + \tan x\right| + C_1$$

or $$I = \frac{1}{2}\left[\sec x \tan x + \log\left|\sec x + \tan x\right|\right] + C$$

(b) $$\int e^x \sin x\,dx$$

Let $$I = \int e^x \sin x\,dx$$

$$= e^x(-\cos x) - \int e^x(-\cos x)\,dx \;= -e^x \cos x + \int e^x \cos x\,dx$$

$$= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x\,dx\right)$$

$\therefore$ $$I = -e^x \cos x + e^x \sin x - 1$$

or

$$2I = -e^x \cos x + e^x \sin x$$

or $$I = \frac{e^x}{2}\left(\sin x - \cos x\right) + C$$

Example 30.33 Evaluate:

$$\int \sqrt{a^2 - x^2}\,dx$$

Solution:

Let $$I = \int \sqrt{a^2 - x^2}\,dx = \int \sqrt{a^2 - x^2} \cdot 1\,dx$$

Integrating by parts only and taking 1 as the second function, we have

$$I = \left(\sqrt{a^2 - x^2}\right)x - \int \frac{1}{2\sqrt{a^2 - x^2}}(-2x) \cdot x\, dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}}dx \quad = x\sqrt{a^2 - x^2} + \int \frac{a^2 - \left(a^2 - x^2\right)}{\sqrt{a^2 - x^2}}dx$$

$$= x\sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}dx - \int \sqrt{a^2 - x^2}\,dx$$

$\therefore$
$$I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) - 1$$

or
$$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

or
$$I = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$$

Similarly,
$$\int \sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

$\therefore$
$$\int \sqrt{a^2 + x^2}\,dx = \frac{x\sqrt{a^2 + x^2}}{2} - \frac{a^2}{2}\log\left|x + x\sqrt{a^2 + x^2}\right| + C$$

Example 30.34 Evaluate:

(a) $\int \sqrt{16x^2 + 25}dx$ (b) $\int \sqrt{16 - x^2}\,dx$ (c) $\int \sqrt{1 + x - 2x^2}\,dx$

Solution:

(a) $$\int \sqrt{16x^2 + 25}dx = 4\int \sqrt{x^2 + \frac{25}{16}}dx = 4\int \sqrt{x^2 + \left(\frac{5}{4}\right)^2}\,dx$$

Using the formula for $\int \sqrt{\left(x^2 + a^2\right)}dx$ we get,

$$\int \sqrt{16x^2 + 25}dx = \left[\frac{x}{2}\sqrt{x^2 + \frac{25}{16}} + \frac{25}{32}\log\left|x + \sqrt{x^2 + \frac{25}{16}}\right|\right] + C$$

$$= \frac{x}{8}\sqrt{16x^2 + 25} + \frac{25}{8}\log\left|4x + \sqrt{16x^2 + 25}\right| + C$$

Notes

(b) Using the formula for $\int \sqrt{(a^2-x^2)}\,dx$ we get,

$$\int\sqrt{16-x^2}\,dx=\int\sqrt{(4)^2-x^2}\,dx=\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\frac{x}{4}+C$$

(c)
$$\int\sqrt{1+x-2x^2}\,dx=\sqrt{2}\int\sqrt{\frac{1}{2}+\frac{x}{2}-x^2}\,dx$$

$$=\sqrt{2}\int\sqrt{\left[\frac{1}{2}-\left(x^2-\frac{x}{2}+\frac{1}{16}\right)+\frac{1}{16}\right]}\,dx$$

$$=\sqrt{2}\int\sqrt{\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}\,dx$$

$$=\sqrt{2}\left[\frac{x-\frac{1}{4}}{2}\sqrt{\frac{9}{16}-\left(x-\frac{1}{4}\right)^2}+\frac{9}{16\times 2}\sin^{-1}\frac{x-\frac{1}{4}}{\frac{3}{4}}\right]+C$$

$$=\sqrt{2}\left[\frac{4x-1}{8}\cdot\frac{1}{\sqrt{2}}\sqrt{1+x-2x^2}+\frac{9}{32}\sin^{-1}\frac{4x-1}{3}\right]+C$$

$$=\frac{4x-1}{8}\sqrt{1+x-2x^2}+\frac{9\sqrt{2}}{32}\sin^{-1}\frac{4x-1}{3}+C$$

CHECK YOUR PROGRESS 30.7

Evaluate:

1. (a) $\int e^x \sec x[1+\tan x]\,dx$ (b) $\int e^x\left[\sec x+\log|\sec x+\tan x|\right]\mathrm{dx}$

2. (a) $\int \frac{x-1}{x^2}e^x\,dx$ (b) $\int e^x\left(\sin^{-1}x-\frac{1}{\sqrt{1-x^2}}\right)dx$

3. $\int e^x\frac{(x-1)}{(x+1)^3}\,dx$ 4. $\int\frac{xe^x}{(x+1)^2}\,dx$

5. $\int\frac{x+\sin x}{1+\cos x}\,dx$ 6. $\int e^x\sin 2x\,dx$

30.8 INTEGRATION BY USING PARTIAL FRACTIONS

By now we are equipped with the various techniques of integration.

But there still may be a case like $\frac{4x+5}{x^2+x-6}$, where the substitution or the integration by parts may not be of much help. In this case, we take the help of another technique called **techmique of integrayion using partial functions.**

Any proper rational fraction $\frac{p(x)}{q(x)}$ can be expressed as the sum of rational functions, each having a single factor of q(x). Each such fraction is known as **partial fraction** and the process of obtaining them is called decomposition or resolving of the given fraction into partial fractions.

For example, $\quad \frac{3}{x+2}+\frac{5}{x-1}=\frac{8x+7}{(x+2)(x-1)}=\frac{8x+7}{x^2+x-2}$

Here $\frac{3}{x+2}, \frac{5}{x-1}$ are called partial fractions of $\frac{8x+7}{x^2+x-2}$.

If $\frac{f(x)}{g(x)}$ is a proper fraction and g (x)can be resolved into real factors then,

(a) corresponding to each non repeated linear factor ax + b, there is a partial fraction of the form

(b) for $(ax+b)^2$ we take the sum of two partial fractions $\cdot$ as

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

For $(ax+b)^3$ we take the sum of three partial fractions as

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$$

and so on.

(c) For non-fractorisable quadratic polynomial ax^2+bx+c there is a partial fraction

$$\frac{Ax+B}{ax^2+bx+c}$$

Therefore, if g (x) is a proper fraction $\frac{f(x)}{g(x)}$ and can be resolved into real factors, $\frac{f(x)}{g(x)}$ can be written in the follwoing form:

Factor in the denominator	corresponding partial fraction
ax+b	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}$
$(ax+b)^3$	$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}$

where A,B,C,D are arbitary constants.

The rational functions which we shall consider for integration will be those whose denominators can be fracted into linear and quadratic factors.

Example 30.35 Evaluate:

$$\int \frac{2x+5}{x^2-x-2}dx$$

Solution: $\frac{2x+5}{x^2-x-2}=\frac{2x+5}{(x-2)(x+1)}$

Let $\frac{2x+5}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$

Multiplyping both sides by $(x-2)(x+1)$,we have

$$2x+5=A(x+1)+B(x-2)$$

Putting x = 2, weget 9 =3A or A = 3

Putting x = -1, we get 3 = -3B or B = -1

substituting these values in(1),we have

$$\frac{2x+5}{(x-2)(x+1)}=\frac{3}{x-2}-\frac{1}{x+1}$$

$$\Rightarrow \quad \int\frac{2x+5}{x^2-x-2}dx=\int\frac{3}{x-2}dx-\int\frac{1}{x+1}dx$$

$$=3\log|x-2|-\log|x+1|+C$$

Example 30.36 Evaluate:

$$\int\frac{x^3+x+1}{x^2-1}dx$$

Solution: $I=\int\frac{x^3+x+1}{x^2-1}dx$

Now $\frac{x^2+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}=x+\frac{2x+1}{(x+1)(x-1)}$

$$\therefore \quad I=\int\left(x+\frac{2x+1}{(x+1)(x-1)}\right)dx$$

Let $\frac{2x+1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$ (2)

$$\Rightarrow \quad 2x+1=A(x-1)+B(x+1)$$

Putting x = 1, we get $B=\frac{3}{2}$

Putting x = -1, we get $A=\frac{1}{2}$

Substituting the values of A and B in (2) and integrating, we have

$$\int\frac{2x+1}{(x^2-1)}dx=\frac{1}{2}\int\frac{1}{(x^2+1)}dx+\frac{3}{2}\int\frac{1}{x-1}dx$$

$$=\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1| \qquad (3)$$

$\therefore$ From (1) and (3), we have

$$I = \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

Example 30.37 Evaluate:

$$\int \frac{8}{(x-2)(x^2+4)}\,dx$$

Solution:

(a) $$\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

(As x^2+4 is not factorisable into linear factors)

Multiplying both sides by $(x+2)(x^2+4)$, we have

$$8 = A(x^2+4) + (Bx+C)(x+2)$$

On comparing the corresponding coeffcients of powers of x on both sides, we get

$$\left.\begin{aligned} 0 &= A+B \\ 0 &= 2B+C \\ 8 &= 4A+2C \end{aligned}\right\} \Rightarrow A=1, B=-1, C=2$$

$$\therefore \quad \int \frac{8}{(x+2)(x^2+4)}\,dx = \int \frac{1}{x+2}\,dx - \int \frac{x-2}{x^2-4}\,dx$$

$$= \int \frac{1}{x+2}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + 2\int \frac{dx}{x^4+4}$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\frac{x}{2} + C$$

Example 30.38 Evaluate:

$$\int \frac{2\sin 2\theta - \cos\theta}{4-\cos^2\theta - 4\sin\theta}\,d\theta$$

Solution:

Let $$\mathrm{I} = \int \frac{2\sin 2\theta - \cos\theta}{4-\cos^2\theta - 4\sin\theta}\,d\theta = \int \frac{(4\sin\theta - 1)\cos\theta\,d\theta}{3+\sin^2\theta - 4\sin\theta}$$

Let $\sin\theta$ = t, then $\cos\theta\, d\theta$ = dt

$$\therefore \qquad I = \int \frac{4t-1}{3+t^2-4t}\,dt$$

Let $\dfrac{4t-1}{3-t^2-4t} = \dfrac{A}{t-3} + \dfrac{B}{t-1}$ Thus $4t-1 = A(t-1)+b(t-3)$

Put $t=1 \; then B = -\dfrac{3}{2}$ Put $t=3 \; then \; A = \dfrac{11}{2}$

$$\therefore \qquad I = \frac{11}{2}\int\left(\frac{1}{t-3}\right)dt - \frac{3}{2}\int\frac{dt}{t-1} = \frac{11}{2}\log|t-3| - \frac{3}{2}\log|t-1| + C$$

$$= \frac{11}{2}\log|\sin\theta-3| - \frac{3}{2}\log|\sin\theta-1| + C$$

$$= -\frac{1}{3}\int\frac{dt}{1+t} + \frac{1}{6}\int\frac{(2t-1)\,dt}{t^2-t+1} + \frac{1}{2}\int\frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log|\tan^2\theta-\tan\theta+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right) + C$$

CHECK YOUR PROGRESS 30.8

Evaluate the following:

1. (a) $\int\sqrt{4x^2-5}\,dx$ (b) $\int\sqrt{x^2+3x}\,dx$ (c) $\sqrt{3-2x-2x^2}\,dx$

2. (a) $\int\dfrac{x+1}{(x-2)(x-3)}\,dx$ (b) $\int\dfrac{x}{x^2-16}\,dx$

3. (a) $\int\dfrac{x^2}{x^2-4}\,dx$ (b) $\int\dfrac{2x^2+x+1}{(x-1)^2(x+2)}\,dx$

MODULE - VIII
Calculus

Notes

4. $\int \frac{x^2+x+1}{(x-1)^3}dx$

5. (a) $\int \frac{\sin x}{\sin 4x}dx$ (b) $\int \frac{1-\cos x}{\cos x(1+\cos x)}dx$

LET US SUM UP

- Integration is the inverse of differentiation
- Standard form of some inddefinite integrals

(a) $\int x^n dx \quad = \frac{x^{n+1}}{n+1}+C\,(n \neq -1)$

(b) $\int \frac{1}{x}dx \quad = \log|x|+C$

(c) $\int \sin x\, ax \quad = -\cos x + C$

(d) $\int \cos x\; dx \quad = \sin x + C$

(e) $\int \sec^2 x\, dx \quad = \tan x + C$

(f) $\int \cos ec^2 x\, dx \quad = -\cot x + C$

(g) $\int \sec x \tan x dx \quad = \sec x + C$

(h) $\int \cos ecx \cot x\, dx \quad = -\cos ecx + C$

(i) $\int \frac{1}{\sqrt{1-x^2}}dx \quad = \sin^{-1} x + C$

(j) $\int \frac{1}{1+x^2}dx \quad = \tan^{-1} x + C$

(k) $\int \frac{1}{x\sqrt{x^2-1}}dx \quad = \sec^{-1} x + C$

(l) $\int e^x dx \quad = e^x + C$

(m) $\int a^x\; dx \quad = \frac{a^x}{\log} + C\,(a > 0\, and\; a \neq 1)$

- Properties of indefinite integrals

(a) $\int [f(x) \pm g(x)]dx \quad = \int f(x)dx \pm \int g(x)dx$

(b) $\int kf(x)\,dx \qquad = k\int f(x)\,dx$

(i) $\int (ax+b)^n dx \qquad = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C \; (n \neq -1)$

(ii) $\int \frac{1}{ax+b}dx \qquad = \frac{1}{a}\log|ax+b| + C$

(iii) $\int \sin(ax+b)\,dx \qquad = \frac{-1}{a}\cos(ax+b) + C$

(iv) $\int \cos(ax+b)\,dx \qquad = \frac{1}{a}\sin(ax+b) + C$

(v) $\int \sec^2(ax+b)\,dx \qquad = \frac{1}{a}\tan(ax+b) + C$

(vi) $\int co\sec^2(ax+b)\,dx \qquad = \frac{1}{a}\cot(ax+b) + C$

(vii) $\int \sec(ax+b)\tan(ax+b)\,dx \qquad = \frac{1}{a}\sec(ax+b) + C$

(viii) $\int co\sec(ax+b)\cot(ax+b)\,dx = \frac{1}{a}co\sec(ax+b) + C$

(ix) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$

(i) $\int \tan x\,dx \qquad = -\log|\cos x| + C = \log|\sec x| + C$

(ii) $\int \cot x\,dx \qquad = \log|\sin x| + C$

(iii) $\int \sec x\,dx \qquad = \log|\sec x + \tan x| + C$

(iv) $\int co\sec x\,dx \qquad = \log|co\sec x - \cot x| + C$

- (i) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(ii) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a-x}{a+x}\right| + C$

(iii) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(iv) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$

Notes

(v) $$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right| + C$$

(vi) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$$

- Integral of the product of two functions

I function $\times$ Integral of II function $-\int$[Derivative of I function $\times$ Integral of II function]dx

- $$\int e^x \left[f(x) + f(x) \right] dx = e^x f(x) + C$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$$

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

$$\int \sqrt{a^2 + x^2}\, dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + C$$

- Rational fractions are of following two types:

(i) Proper, where degree of variable of numerator $<$ denominator.

(ii) Improper, where degree of variable of numerator $\geq$ denominator.

- If g (x) is a proper fraction $\frac{f(x)}{g(x)}$ can be resolved into real factors, then $\frac{f(x)}{g(x)}$ can be written in the following form :

Factors in denominator	**Corresponding partial fraction**
a x + b	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
$(ax+b)^3$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$
$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}$

where A,B,C, D are arbitary constants.

SUPPORTIVE WEB SITES

http://www.bbc.co.uk/education/asguru/maths/12methods/04integration/index.shtml

http://en.wiktionary.org/wiki/integration

http://www.sosmath.com/calculus/integration/byparts/byparts....

TERMINAL EXERCISE

Integrate the following functions w.r.t.x:

1. $\dfrac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}$
2. $\sqrt{1+\sin 2x}$
3. $\dfrac{\cos 2x}{\cos^2 x\sin^2 x}$
4. $(\tan x-\cot x)^2$
5. $\dfrac{4}{1+x^2}-\dfrac{1}{\sqrt{1-x^2}}$
6. $\dfrac{2\sin^2 x}{1+\cos 2x}$
7. $\dfrac{2\cos^2 x}{1-\cos 2x}$
8. $\left(\sin\dfrac{x}{2}+\cos\dfrac{x}{2}\right)^2$
9. $\left(\cos\dfrac{x}{2}-\sin\dfrac{x}{2}\right)^2$
10. $\cos(7x-\pi)$
11. $\sin(3x+4)$
12. $\cos^2(2x+b)$
13. $\displaystyle\int\frac{dx}{\sin x-\cos x}$
14. $\displaystyle\int\frac{1}{(1+x^2)\tan^{-1}x}dx$
15. $\displaystyle\int\frac{\cos ec}{\log\left(\tan\dfrac{x}{2}\right)}dx$
16. $\displaystyle\int\frac{\cot}{3+4\log\sin x}dx$
17. $\displaystyle\int\frac{dx}{\sin 2x\log\tan x}$
18. $\displaystyle\int\frac{e^x+1}{e^x-1}dx$
19. $\displaystyle\int\sec^4 x\tan x dx$
20. $\displaystyle\int e^2\sin e^x dx$
21. $\displaystyle\int\frac{x\,dx}{\sqrt{2x^2+3}}$
22. $\displaystyle\int\frac{\sec^2 x}{\sqrt{\tan x}}dx$
23. $\displaystyle\int\sqrt{25-9x^2}dx$
24. $\displaystyle\int\sqrt{2ax-x^2dx}$
25. $\displaystyle\int\sqrt{3x^2+4dx}$
26. $\displaystyle\int\sqrt{1+9x^2}dx$
27. $\displaystyle\int\frac{x^2dx}{\sqrt{x^2-a^2}}$
28. $\displaystyle\int\frac{dx}{\sin^2 x+4\cos^2 x}$
29. $\displaystyle\int\frac{dx}{2+\cos x}$
30. $\displaystyle\int\frac{dx}{x^2-6x+13}$
31. $\displaystyle\int\frac{dx}{1+3\sin^2 x}$
32. $\displaystyle\int\frac{x^2}{x^2-a^2}dx$
33. $\displaystyle\int\frac{dx}{x\sqrt{9+x^4}}$
34. $\displaystyle\int\frac{\sin}{\sin 3x}dx$
35. $\displaystyle\int\frac{dx}{1-4\cos^2 x}$
36. $\displaystyle\int\sec^2(ax+b)dx$

Notes

37. $\int \frac{dx}{x(2+\log x)}$

38. $\int \frac{x^5}{1+x^6}dx$

39. $\int \frac{\cos x-\sin x}{\sin x+\cos x}dx$

40. $\int \frac{\cot x}{\log \sin x}dx$

41. $\int \frac{\sec^2 x}{a+b\tan x}dx$

42. $\int \frac{\sin x}{1+\cos}dx$

43. $\int \cos^2 x\,dx$

44. $\int \sin^3 x\,dx$

45. $\int \sin 5x \sin 3x\,dx$

46. $\int \sin^2 x\cos^3 x\,dx$

47. $\int \sin^4 x\,dx$

48. $\int \frac{1}{1+\sin x}dx$

49. $\int \tan^3 x\,dx$

50. $\int \frac{\cos x-\sin x}{1+\sin 2x}dx$

51. $\int \frac{\cos ec^2 x}{1+\cot x}dx$

52. $\int \frac{1+x+\cos 2x}{x^2+\sin 2x+2x}dx$

53. $\int \frac{\sec\theta \cos ec\theta\, d\theta}{\log \tan\theta}$, 54. $\int \frac{\cot\theta\, d\theta}{\log \sin\theta}$

55. $\int \frac{dx}{1+4x^2}$

56. $\int \frac{1-\tan\theta}{1+\tan\theta}d\theta$

57. $\int \frac{1}{x^2}e^{\frac{-1}{x}}dx$

58. $\int \frac{\sin x\cos x\,dx}{a^2\sin^2 x+b^2\cos^2 x}$

59. $\int \frac{dx}{\sin x+\cos x}$

60. $\int e^x\left(\cos^{-1}x-\frac{1}{\sqrt{1-x^2}}\right)dx$

61. $\int e^x\left(\frac{\sin x+\cos x}{\cos^2 x}\right)dx$

62. $\int \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}dx$

63. $\int \cos\left[2\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]dx$

64. $\int \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}dx$

65. $\int \sqrt{x}\log x\,dx$

66. $\int e^x(1+x)\log(xe^x)\,dx$

67. $\int \frac{\log x}{(1+x)^2}dx$

68. $\int e^x\sin^2 x\,dx$

69. $\int \cos(\log x)\,dx$

70. $\int \log(x+1)\,dx$

71. $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

72. $\int \frac{\sin\theta\cos\theta}{\cos^2\theta-\cos\theta-2}dx$

73. $\int \frac{dx}{x(x^5+1)}$

74. $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}dx$

75. $\int \frac{\log x}{x(1+\log x)(2+\log x)}dx$

76. $\int \frac{dx}{1-e^x}$

ANSWERS

CHECK YOUR PROGRESS 30.1

1. $\frac{2}{7}x^{\frac{7}{2}}+1, \frac{2}{7}x^{\frac{7}{2}}+2, \frac{2}{7}x^{\frac{7}{2}}+3, \frac{2}{7}x^{\frac{7}{2}}+4, \frac{2}{7}x^{\frac{7}{2}}+5$

2. (a) $\frac{x^6}{6}+C$ (b) $\sin x+C$ (c) 0

3. (a) $\frac{x^7}{7}+C$ (b) $\frac{1}{6x^6}+C$ (c) $\log|x|+C$

   (d) $\frac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)}+C$ (e) $\frac{3}{4}x^{\frac{4}{3}}+C$ (f) $\frac{-1}{8x^8}+C$

   (g) $2\sqrt{x}+C$ (h) $9x^{\frac{1}{9}}+C$

4. (a) $-\cos es\theta+C$ (b) $\sec\theta+C$

   (c) $\tan\theta+C$ (d) $-\cot\theta+C$

CHECK YOUR PROGRESS 30.2

1. (a) $\frac{x^2}{2}+\frac{1}{2}x+C$ (b) $-x+\tan^{-1}x+C$

   (c) $x^{10}-\frac{2}{3}x^{\frac{3}{2}}+2\sqrt{x}+C$ (d) $-\frac{1}{x^5}-\frac{3}{4x^4}+\frac{2}{3x^3}+\frac{7}{x}-8x+C$

   (e) $\frac{x^3}{3}-x-\tan^{-1}x+C$ (f) $\frac{x^2}{2}+4x+4\log x+C$

2. (a) $\frac{1}{2}\tan x+C$ (b) $\tan x-x+C$

   (c) $-2\cos ec\,x+C$ (d) $-\frac{1}{2}\cot x+C$

   (e) $-\sec x+C$ (f) $-\cot x+\cos ec\,x+C$

3. (a) $\sqrt{2}\sin x + C$ (b) $-\sqrt{2}\cos x + C$

(c) $-\frac{1}{2}\cot x + C$

4. (a) $\frac{2}{3}(x+2)^{\frac{3}{2}} + C$

CHECK YOUR PROGRESS 30.3

1. (a) $\frac{1}{5}\cos(4-5x) + C$ (b) $\frac{1}{3}\tan(2+3x) + C$

(c) $\log\left|\sec\left(x+\frac{\pi}{4}\right) + \tan\left(x+\frac{\pi}{4}\right)\right| + C$

(d) $\frac{1}{4}\sin(4x+5) + C$ (e) $\frac{1}{3}\sec(3x+5) + C$

(f) $-\frac{1}{5}\cos ec(3+5x) + C$

2. (a) $\frac{1}{12(3-4x)^3} + C$ (b) $\frac{1}{5}(x+1)^5 + C$

(c) $-\frac{1}{77}(4-7x)^{11} + C$ (d) $\frac{1}{16}(4x-5)^4 + C$

(e) $\frac{1}{3}\log|3x-5| + C$ (f) $-\frac{2}{9}\sqrt{5-9x} + C$

(g) $\frac{1}{6}(2x+1)^3 + C$ (h) $\log|x+1| + C$

3. (a) $\frac{1}{2}e^{2x+1} + C$ (b) $-\frac{1}{8}e^{3-8x} + C$

(c) $-\frac{1}{4e^{(7+4x)}} + C$

4. (a) $\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$ (b) $\frac{1}{32}\left(-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right) + C$

(c) $\frac{1}{2}\left(-\frac{\cos 7x}{7} - \cos x\right) + C$ (d) $\frac{1}{2}\left(\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right) + C$

CHECK YOUR PROGRESS 30.4

1. (a) $\frac{1}{6}\log\left|3x^2-2\right|+C$ (b) $\log\left|x^2+x+1\right|+C$

(c) $\log\left|x^2+9x+30\right|+C$ (d) $\frac{1}{3}\log\left|x^3+3x+3\right|+C$

(e) $\log\left|x^2+x-5\right|+C$ (f) $2\log\left|5+\sqrt{x}\right|+C$

(g) $\log\left|8+\log x\right|+C$

2. (a) $\frac{1}{b}\log\left|a+be^x\right|+C$ (b) $\tan^{-1}\left(e^x\right)+C$

CHECK YOUR PROGRESS 30.5

1. (a) $x+\frac{3}{2}\log\left|\frac{x-3}{x+3}\right|+C$ (b) $\tan^{-1}\left(e^x\right)+C$

(c) $\frac{1}{2}\tan^{-1}\left(x^2\right)+C$ (d) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$

(e) $\frac{1}{2}\tan^{-1}\left(2\tan x\right)+C$ (f) $\sin^{-1}\left(\frac{x+1}{2}\right)+C$

(g) $\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$ (h) $\sin^{-1}\left(\frac{x+2}{3}\right)+C$

(i) $\frac{1}{2\sqrt{3}}\sec^{-1}\frac{x}{2}+C$ (j) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2\theta-1}{\sqrt{2}\tan\theta}\right)+C$

(k) $\log\left|e^x+\sqrt{1+e^{2x}}\right|+C$ (l) $\sin^{-1}x-\sqrt{1-x^2}+C$

(m) $\sin^{-1}\left(\frac{x-a}{a}\right)+C$ (n) $\frac{1}{4}\sin^{-1}\left(\frac{4}{3}x^3\right)+C$

(o) $\sqrt{x^2+1}+\log\left|x+\sqrt{x^2+1}\right|+C$

(p) $\frac{1}{2}\log\left|\frac{2x+\sqrt{9+4x^2}}{2}\right|+C$

(q) $-\frac{1}{2}\log\left|2\cos\theta+\sqrt{4\cos^2\theta-1}\right|+C$

(r) $\log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C$

(s) $\tan^{-1}\left(\frac{x+2}{1}\right)+C$ (t) $\frac{1}{4}\log\left|x+\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right|+C$

CHECK YOUR PROGRESS 30.6

1. (a) $-x\cos x+\sin+C$

(b) $\frac{1}{2}\left(1+x^2\right)\sin 2x+\frac{x\cos 2x}{2}-\frac{\sin 2x}{4}+C$

(c) $\frac{-x\cos 2x}{2}+\frac{1}{2}\frac{\sin 2x}{2}+C$

2. (a) $x\tan x-\log\left|\sec x\right|-x+C$

(b) $\frac{1}{6}x^3-\frac{1}{4}x^2\sin 2x-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$

3. (a) $\frac{x^4\log 2x}{4}-\frac{x^4}{16}+C$ (b) $\left(x-\frac{x^3}{3}\right)\log x-x+\frac{x^3}{9}+C$

(c) $x\left(\log x\right)^2-2x\log x+2x+C$

4. (a) $\frac{x^{1-n}}{1-n}\log x-\frac{x^{1-n}}{\left(1-n\right)^2}+C$ (b) $\log x.\left[\log\left(\log x\right)-1\right]+C$

5. (a) $e^{3x}\left[\frac{x^2}{3}-\frac{2x}{9}+\frac{2}{27}\right]+C$ (b) $x\frac{e^{4x}}{4}-x\frac{e^{4x}}{16}+C$

6. (a) $\frac{x^2}{2}\left[\left(\log x\right)^2-\log x+\frac{1}{2}\right]+C$

7. (a) $x\sec^{-1}x-\log\left|x+\sqrt{x^2-1}\right|+C$

(b) $\frac{x^2}{2}\cot^{-1}x+\frac{x}{2}+\frac{1}{2}\cot^{-1}x+C$

CHECK YOUR PROGRESS 30.7

1. (a) $e^x \sec x0 + C$ (b) $e^x \log|\sec x + \tan x| + C$

2. (a) $\frac{1}{x}e^x + C$ (b) $e^x \sin^{-1} x + C$

3. $\frac{e^x}{(1+x)^2} + C$

4. $\frac{e^X}{1+x} + C$

5. $x \tan\frac{x}{2} + C$

6. $\frac{1}{5}e^x(\sin 2x - 2\cos 2x) + C$

CHECK YOUR PROGRESS 30.8

1. (a) $x\sqrt{x^2 - \frac{5}{4}} - \frac{5}{4}\log\left|x + \sqrt{x^2 - \frac{5}{4}}\right| + C$

   (b) $\frac{(2x+3)}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^{2+}3x}\right| + C$

   (c) $\frac{1}{4}(2x+1)\sqrt{3-2x-2x^2} + \frac{7}{4\sqrt{2}}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C$

2. (a) $4\log|x-3| - 3\log|x-2| + C$

   (b) $\frac{1}{2}\log|x-4| + \log|x+4| + C$

3. (a) $\frac{x^2}{2} - 2\left[\log|x-2|| + \log||x+2|\right] + c$

   (b) $\frac{11}{9}\log|x-1| + \frac{7}{9}\log(x+2) - \frac{4}{3(x-1)} + C$

4. $\log|x-1| - \frac{3}{(x-1)} - \frac{3}{2(x-1)^2} + C$

5. (a) $\frac{1}{8}\log|1-\sin x| - \frac{1}{8}|1+\sin x|$

   $-\frac{1}{4\sqrt{2}}\log\left|1-\sqrt{2}\sin x\right| + \frac{1}{4\sqrt{2}}\log\left|1+\sqrt{2}\sin x\right| + C$

Notes

(b) $\log\left|\sec x+\tan x\right|-2\tan\dfrac{x}{2}+C$

TERMINAL EXERCISE

1. $\sec x-\operatorname{cosec} x+C$
2. $\sin x-\cos x+C$
3. $-\cot x-\tan x+C$
4. $\tan x-\cot x-4x+C$
5. $4\tan^{-1}x-\sin^{-1}x+C$
6. $\tan x-x+C$
7. $-\cot x-x+C$
8. $x-\cos x+C$
9. $x+\cos x+C$
10. $\dfrac{\sin(7x-\pi)}{7}+C$
11. $\dfrac{-\cos(3x+4)}{3}+C$
12. $\dfrac{\tan(2x+b)}{2}+C$
13. $\dfrac{1}{\sqrt{2}}\log\left|\operatorname{cosec}\left(x-\dfrac{\pi}{4}\right)-\cot\left(x-\dfrac{\pi}{4}\right)\right|+C$
14. $\log\left|\tan^{-1}x\right|+C$
15. $\log\left|\log\tan\dfrac{x}{2}\right|+C$
16. $\dfrac{1}{4}\log\left|3+4\log\sin x\right|+C$
17. $\dfrac{1}{2}\log\left|\log\tan x\right|+C$
18. $2\log\left|e^{\frac{x}{2}}-e^{\frac{-x}{2}}\right|+C$
19. $\dfrac{1}{4}\sec^4 x+C$
20. $-\cos e^{x}+C$
21. $\dfrac{\sqrt{2x^2+3}}{2}+C$
22. $2\sqrt{\tan x}+C$
23. $\dfrac{1}{6}x\sqrt{(25-9x^2)}+\dfrac{25}{6}\sin^{-1}\left(\dfrac{3}{5}x\right)+C$
24. $\dfrac{1}{2}(x-a)\sqrt{2ax-x^2}+\dfrac{1}{2}a^2\sin^{-1}\left(\dfrac{x-a}{a}\right)+C$
25. $\dfrac{x\sqrt{3x^2+4}}{2}+\dfrac{2}{\sqrt{3}}\log\left|\dfrac{\sqrt{3}x+\sqrt{x^2+4}}{2}\right|+C$

26. $\dfrac{x\sqrt{9x^2+1}}{2}+\dfrac{1}{6}\log\left|3x+\sqrt{1+9x^2}\right|+C$

27. $\left[\dfrac{1}{2}x\sqrt{x^2-a^2}+\dfrac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right|\right]+C$

28. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{\tan x}{2}\right)+C$

29. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left[\dfrac{\tan\left(\dfrac{x}{2}\right)}{\sqrt{3}}\right]+C$

30. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x-3}{2}\right)+C$

31. $\dfrac{1}{2}\tan^{-1}\left(2\tan x\right)+C$

32. $x+\dfrac{a}{2}\log\left|\dfrac{x-a}{x+a}\right|+C$

33. $\dfrac{1}{12}\log\left|\dfrac{\sqrt{9+x^4}-3}{\sqrt{9+x^4}+3}\right|+C$

34. $\dfrac{2}{2\sqrt{3}}\log\left|\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|+C$

35. $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{\tan x-\sqrt{2}}{\tan x+\sqrt{2}}\right|+C$

36. $\dfrac{1}{a}\tan\left(ax+b\right)+C$

37. $\log\left|\left(2+\log x\right)\right|+C$

38. $\dfrac{1}{6}\log\left(1+x^6\right)+C$

39. $\log\left|\sin x+\cos x\right|+C$

40. $\log\left|\log\left(\sin x\right)\right|+C$

41. $\dfrac{1}{b}\log\left|a+b\tan x\right|+C$

42. $-\log\left|1+\cos x\right|+C$

43. $\dfrac{1}{2}\dfrac{\sin 2x}{2}+\dfrac{1}{2}x+C$

44. $-\cos x+\dfrac{\cos^3 x}{3}+C$

45. $\dfrac{1}{2}\dfrac{\sin 2x}{2}-\dfrac{1}{2}\dfrac{\sin 8x}{8}+C$

46. $\dfrac{1}{3}\sin^3 x-\dfrac{\sin^5 x}{5}+C$

47. $\dfrac{1}{32}\left[12x-8\sin 2x+\sin 4x\right]+C$

48. $\tan x-\sec x+C$

49 $\dfrac{\tan^2 x}{2}+\log\left|\cos x\right|+C$.

50. $\dfrac{-1}{\cos x+\sin x}+C$

51. $\log\left|\dfrac{1}{1+\cot x}\right|+C$

Notes

52. $\frac{1}{2}\log\left|x^2+\sin 2x+2x\right|+C$

53 $\log\left|\tan\theta\right|+C$.

54. $\log\left|\log\sin\theta\right|+C$

55. $\frac{1}{2}\tan^{-1}2x$

56. $\log\left|\cos\theta+\sin\theta\right|+C$

57. $e^{\frac{1}{x}}+C$

58. $\frac{1}{2\left(a^2-b^2\right)}\log\left|a^2\sin^2 x+b^2\cos^2 x\right|+C$

59. $\frac{1}{\sqrt{2}}\log\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|+C$

60. $e^x\cos^{-1}x+C$

61. $e^x\sec x+C$

62. $\frac{1}{4}x^2+C$

63. $-\frac{1}{2}x^2+C$

64. $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}+\frac{1}{2}\log\left|1-x^2\right|+C$

65. $\frac{2}{3}x^{\frac{3}{2}}\left(\log x-\frac{2}{3}\right)+C$

66. $xe^x\left[\log\left(xe^x\right)-1\right]+C$

67. $-\frac{1}{1+x}\log|x|+\log|x|-\log|x+1|+c$

68. $\frac{1}{2}e^x-\frac{e^x}{10}\left(2\sin 2x+\cos 2x\right)+C$

69. $\frac{x}{2}\left[\cos\left(\log x\right)+\sin\left(\log x\right)\right]+C$

70. $x\log|x+1|-x+\log|x+1|+C$

71. $\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$

72. $-\frac{2}{3}\log|\cos\theta-2|-\frac{1}{3}\log|\cos\theta+1|+C$

73. $\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+C$

74. $\frac{1}{3\sqrt{2}}\left[\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\tan^{-1}\left(\sqrt{2}x\right)\right]+C$

75. $\log\left|\frac{\left(2+\log x\right)^2}{1+\log x}\right|+C$

76. $\log\left|\frac{e^x}{1-e^x}\right|+C$

31

DEFINITE INTEGRALS

In the previous lesson we have discussed the anti-derivative, i.e., integration of a function.The very word integration means to have some sort of summation or combining of results.

Now the question arises : Why do we study this branch of Mathematics? In fact the integration helps to find the areas under various laminas when we have definite limits of it. Further we will see that this branch finds applications in a variety of other problems in Statistics, Physics, Biology, Commerce and many more.

In this lesson, we will define and interpret definite integrals geometrically, evaluate definite integrals using properties and apply definite integrals to find area of a bounded region.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the definite integral as a limit of sum;
- evaluate a given definite integral using above definition;
- state fundamental theorem of integral calculus;
- state and use the following properties for evaluating definite integrals :

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (ii) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

(iii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vi) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x)$

$= 0 \text{ if } f(2a-x) = -f(x)$

MODULE - VIII
Calculus

Notes

(vii) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if f is an even function of x

= 0 if f is an odd function of x.

- apply definite integrals to find the area of a bounded region.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of integration
- Area of a bounded region

31.1 DEFINITE INTEGRAL AS A LIMIT OF SUM

In this section we shall discuss the problem of finding the areas of regions whose boundary is not familiar to us. (See Fig. 31.1)

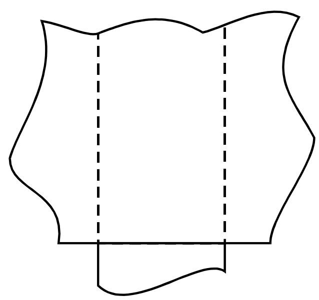


Fig. 31.1

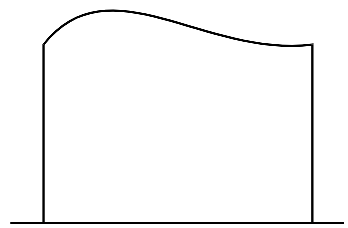


Fig. 31.2

Let us restrict our attention to finding the areas of such regions where the boundary is not familiar to us is on one side of x-axis only as in Fig. 31.2.

This is because we expect that it is possible to divide any region into a few subregions of this kind, find the areas of these subregions and finally add up all these areas to get the area of the whole region. (See Fig. 31.1)

Now, let f (x) be a continuous function defined on the closed interval [a, b]. For the present, assume that all the values taken by the function are non-negative, so that the graph of the function is a curve above the x-axis (See. Fig.31.3).

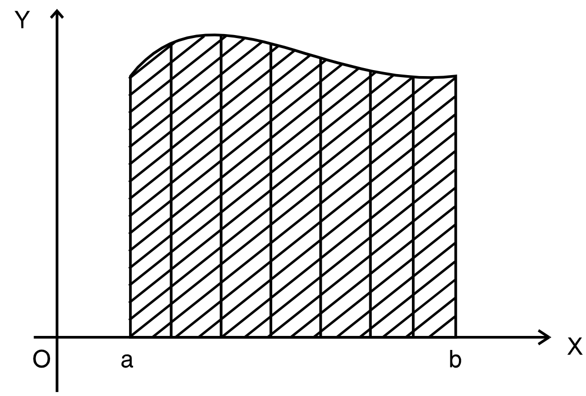


Fig. 31.3

Consider the region between this curve, the x-axis and the ordinates $x = a$ and $x = b$, that is, the shaded region in Fig.31.3. Now the problem is to find the area of the shaded region.

In order to solve this problem, we consider three special cases of $f(x)$ as rectangular region, triangular region and trapezoidal region.

The area of these regions $=$ base $\times$ average height

In general for any function $f(x)$ on $[a, b]$

Area of the bounded region (shaded region in Fig. 31.3) = base $\times$ average height

The base is the length of the domain interval $[a, b]$. The height at any point x is the value of $f(x)$ at that point. Therefore, the average height is the average of the values taken by f in $[a, b]$. (This may not be so easy to find because the height may not vary uniformly.) Our problem is how to find the average value of f in $[a,b]$.

31.1.1 Average Value of a Function in an Interval

If there are only finite number of values of f in $[a,b]$, we can easily get the average value by the formula.

$$\text{Average value of f in } [a, b] = \frac{\text{Sum of the values of f in } [a, b]}{\text{Numbers of values}}$$

But in our problem, there are infinite number of values taken by f in $[a, b]$. How to find the average in such a case? The above formula does not help us, so we resort to estimate the average value of f in the following way:

First Estimate : Take the value of f at 'a' only. The value of f at a is $f(a)$. We take this value, namely $f(a)$, as a rough estimate of the average value of f in $[a,b]$.

Average value of f in $[a, b]$ (first estimate) $= f(a)$ (i)

Second Estimate : Divide $[a, b]$ into two equal parts or sub-intervals.

Let the length of each sub-interval be h, $h = \dfrac{b-a}{2}$.

Take the values of f at the left end points of the sub-intervals. The values are $f(a)$ and $f(a+h)$ (Fig. 31.4)

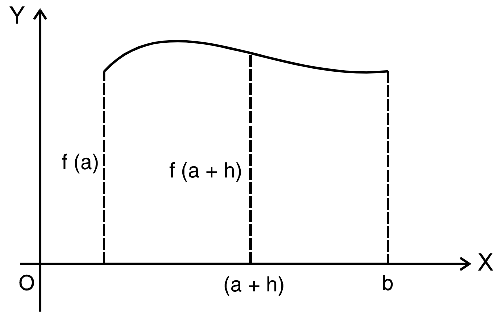


Fig. 31.4

Take the average of these two values as the average of f in [a, b].

Average value of f in [a, b] (Second estimate)

$$= \frac{f(a) + f(a+h)}{2}, \quad h = \frac{b-a}{2} \qquad \text{(ii)}$$

This estimate is expected to be a better estimate than the first.

Proceeding in a similar manner, divide the interval [a, b] into n subintervals of length h

(Fig. 31.5), $h = \frac{b-a}{n}$

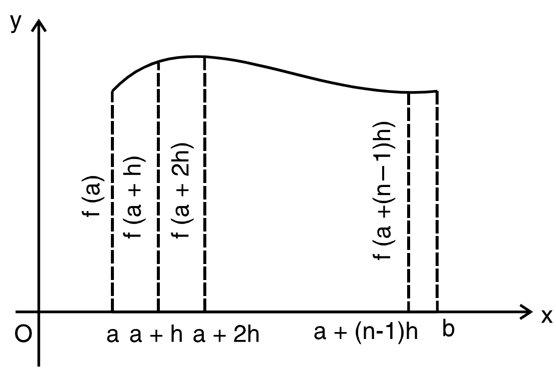


Fig. 31.5

Take the values of f at the left end points of the n subintervals.

The values are f (a), f (a + h),......,f [a + (n-1) h]. Take the average of these n values of f in [a, b].

Average value of f in [a, b] (nth estimate)

$$= \frac{f(a) + f(a+h) + \dots\dots\dots + f(a + (n-1)h)}{n}, \quad h = \frac{b-a}{n} \qquad \text{(iii)}$$

For larger values of n, (iii) is expected to be a better estimate of what we seek as the average value of f in [a, b]

Thus, we get the following sequence of estimates for the average value of f in [a, b]:

$$f(a)$$

$$\frac{1}{2}\left[f(a)+f(a+h)\right], \qquad h=\frac{b-a}{2}$$

$$\frac{1}{3}\left[f(a)+f(a+h)+f(a+2h)\right], \qquad h=\frac{b-a}{3}$$

....

....

$$\frac{1}{n}\left[f(a)+f(a+h)+\text{........}+f\{a+(n-1)h\}\right], \; h=\frac{b-a}{n}$$

As we go farther and farther along this sequence, we are going closer and closer to our destination, namely, the average value taken by f in [a, b]. Therefore, it is reasonable to take the limit of these estimates as the average value taken by f in [a, b]. In other words,

Average value of f in [a, b]

$$\lim_{n\to\infty}\frac{1}{n}\left\{f(a)+f(a+h)+f(a+2h)+\text{......}+f\left[a+(n-1)h\right]\right\},$$

$$h=\frac{b-a}{n} \qquad \text{(iv)}$$

It can be proved that this limit exists for all continuous functions f on a closed interval [a, b].

Now, we have the formula to find the area of the shaded region in Fig. 31.3, The base is $(b-a)$ and the average height is given by (iv). The area of the region bounded by the curve f (x), x-axis, the ordinates x = a and x = b

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left\{f(a)+f(a+h)+f(a+2h)+\text{......}+f\left[a+(n-1)h\right]\right\},$$

$$\lim_{n\to 0}\frac{1}{n}\left[f(a)+f(a+h)+\text{........}+f\{a+(n-1)h\}\right], h=\frac{b-a}{n} \qquad \text{(v)}$$

We take the expression on R.H.S. of (v) as the definition of a **definite integral.** This integral is denoted by

$$\int_a^b f(x)\,dx$$

read as integral of f (x) from a to b'. The numbers a and b in the symbol $\int_a^b f(x)\,dx$ are called respectively the lower and upper limits of integration, and f (x) is called the integrand.

Note : In obtaining the estimates of the average values of f in [a, b], we have taken the left end points of the subintervals. Why left end points?

Why not right end points of the subintervals? We can as well take the right end points of the subintervals throughout and in that case we get

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a+h)+f(a+2h)+......+f(b)\},\quad h=\frac{b-a}{n}$$

$$= \lim_{h\to 0} h\left[f(a+h)+f(a+2h)+......+f(b)\right] \qquad \text{(vi)}$$

Example 31.1 Find $\int_1^2 x\;dx$ as the limit of sum.

Solution : By definition,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+........+f\{a+(n-1)h\}\right],$$

$$h=\frac{b-a}{n}$$

Here a = 1, b = 2, f (x) = x and $h=\frac{1}{n}$.

$$\therefore \qquad \int_1^2 x\,dx = \lim_{n\to\infty}\frac{1}{n}\left[f(1)+f\left(1+\frac{1}{n}\right)+........+f\left(1+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{1}{n}\right)+\left(1+\frac{2}{n}\right)........+\left(1+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\underbrace{1+1+......+1}_{\text{n times}}+\left(\frac{1}{n}+\frac{2}{n}+........+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\left(1+2+......+(n-1)\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n+\frac{(n-1).n}{n.2}\right]$$

$$\left[\text{Since } 1+2+3+....+(n-1)=\frac{(n-1).n}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{3n-1}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{3}{2}-\frac{1}{2n}\right]=\frac{3}{2}$$

Example 31.2 Find $\int_0^2 e^x dx$ as limit of sum.

Solutions : By definition

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ... + f\{a + (n-1)h\}\right]$$

where $\quad h = \dfrac{b-a}{n}$

Here $a = 0, b = 2, f(x) = e^x$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$$\therefore \qquad \int_0^2 e^x dx = \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + ... + f(n-1)h\right]$$

$$= \lim_{h\to 0} h\left[e^0 + e^h + e^{2h} + ... + e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} h\left[e^0\left(\frac{\left(e^h\right)^n - 1}{e^h - 1}\right)\right]$$

$$\left[\text{Since} \quad a + ar + ar^2 + ... + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right)\right]$$

$$= \lim_{h\to 0} h\left[\frac{e^{nh} - 1}{e^h - 1}\right] = \lim_{h\to 0} \frac{h}{h}\left[\frac{e^2 - 1}{\left(\dfrac{e^h - 1}{h}\right)}\right] \qquad (\because nh = 2)$$

$$= \lim_{h\to 0} \frac{e^2 - 1}{\dfrac{e^h - 1}{h}} = \frac{e^2 - 1}{1}$$

$$= e^2 - 1 \qquad \left[\because \lim_{h\to 0} \frac{e^h - 1}{h} = 1\right]$$

In examples 31.1 and 31.2 we observe that finding the definite integral as the limit of sum is quite difficult. In order to overcome this difficulty we have the fundamental theorem of integral calculus which states that

Theorem 1 : If f is continuous in [a, b] and F is an antiderivative of f in [a, b] then

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad ......(1)$$

The difference F (b) – F (a) is commonly denoted by $\left[F(x)\right]_a^b$ so that (1) can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \text{ or } \left[F(x)\right]_a^b$$

In words, the theorem tells us that

$$\int_a^b f(x)\,dx = \text{(Value of antiderivative at the upper limit b)}$$
$$-\text{(Value of the same antiderivative at the lower limit a)}$$

Example 31.3 Evaluate the following

(a) $\int_0^{\frac{\pi}{2}} \cos x\,dx$ (b) $\int_0^2 e^{2x}\,dx$

Solution : We know that

$$\int \cos x\,dx = \sin x + c$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

(b) $$\int_0^2 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^2, \qquad \left[\because \int e^x dx = e^x\right]$$

$$= \left(\frac{e^4 - 1}{2}\right)$$

Theorem 2 : If f and g are continuous functions defined in [a, b] and c is a constant then,

(i) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

(ii) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(iii) $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Example 31.4 Evaluate $\int_0^2 \left(4x^2 - 5x + 7\right)dx$

Solution : $\int_0^2 \left(4x^2 - 5x + 7\right)dx = \int_0^2 4x^2dx - \int_0^2 5x\,dx + \int_0^2 7\,dx$

$$= 4\int_0^2 x^2 dx - 5\int_0^2 x dx + 7\int_0^2 1\, dx$$

$$= 4.\left[\frac{x^3}{3}\right]_0^2 - 5\left[\frac{x^2}{2}\right]_0^2 + 7[x]_0^2$$

$$= 4.\left(\frac{8}{3}\right) - 5\left(\frac{4}{2}\right) + 7(2)$$

$$= \frac{32}{3} - 10 + 14$$

$$= \frac{44}{3}$$

CHECK YOUR PROGRESS 31.1

1. Find $\int_0^5 (x+1)\, dx$ as the limit of sum.
2. Find $\int_{-1}^1 e^x \, dx$ as the limit of sum.
3. Evaluate (a) $\int_0^{\frac{\pi}{4}} \sin x \, dx$ (b) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\, dx$

   (c) $\int_0^1 \frac{1}{1+x^2} dx$ (d) $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

31.2 EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

The principal step in the evaluation of a definite integral is to find the related indefinite integral. In the preceding lesson we have discussed several methods for finding the indefinite integral. One of the important methods for finding indefinite integrals is the method of substitution. When we use substitution method for evaluation the definite integrals, like

$$\int_2^3 \frac{x}{1+x^2} dx, \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx,$$

the steps could be as follows :

(i) Make appropriate substitution to reduce the given integral to a known form to integrate. Write the integral in terms of the new variable.

(ii) Integrate the new integrand with respect to the new variable.

(iii) Change the limits accordingly and find the difference of the values at the upper and lower limits.

Note : If we don't change the limit with respect to the new variable then after integrating resubstitute for the new variable and write the answer in original variable. Find the values of the answer thus obtained at the given limits of the integral.

Example 31.5 Evaluate the following :

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta$ (c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$

Solution : (a) Let $\cos x = t$ then $\sin x\, dx = -\,dt$

When $x = 0$, $t = 1$ and $x = \frac{\pi}{2}, t = 0$. As x varies from 0 to $\frac{\pi}{2}$, t varies from 1 to 0.

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 \frac{1}{1+t^2}\,dt = -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

(b) $$I = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\left(\sin^2\theta+\cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1-2\sin^2\theta\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta\, d\theta}{1-2\sin^2\theta\left(1-\sin^2\theta\right)}$$

Let $\sin^2\theta = t$

Then $2\sin\theta\cos\theta\, d\theta = dt$ i.e. $\sin 2\theta\, d\theta = dt$

When $\theta = 0, t = 0$ and $\theta = \frac{\pi}{2}, t = 1$. As θ varies from 0 to $\frac{\pi}{2}$, the new variable t varies from 0 to 1.

$$\therefore \qquad I = \int_0^1 \frac{1}{1 - 2t(1-t)}\, dt = \int_0^1 \frac{1}{2t^2 - 2t + 1} dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{t^2 - t + \frac{1}{4} + \frac{1}{4}}\, dt \quad I = \frac{1}{2}\int_0^1 \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\, dt$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1 = \left[\tan^{-1} 1 - \tan^{-1}(-1)\right]$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(c) We know that $\cos x = \dfrac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\cos x}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{5 + \dfrac{4\left(1 - \tan^2\left(\frac{x}{2}\right)\right)}{\left(1 + \tan^2\left(\frac{x}{2}\right)\right)}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{9 + \tan^2\left(\frac{x}{2}\right)}\, dx \qquad (1)$$

Let $\qquad \tan\frac{x}{2} = t$

Then $\qquad \sec^2\frac{x}{2} dx = 2dt$ when $x = 0, \quad t = 0$, when $x = \frac{\pi}{2}, \; t = 1$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\cos x} dx = 2\int_0^1 \frac{1}{9 + t^2}\, dt \qquad \text{[From (1)]}$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1 = \frac{2}{3}\left[\tan^{-1}\frac{1}{3}\right]$$

31.3 SOME PROPERTIES OF DEFINITE INTEGRALS

The definite integral of f (x) between the limits a and b has already been defined as

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ Where } \frac{d}{dx}[F(x)] = f(x),$$

where a and b are the lower and upper limits of integration respectively. Now we state below some important and useful properties of such definite integrals.

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ (ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ where $a<c<b$.

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(vi) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vii) $$\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

(viii) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function of } x \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function of } x \end{cases}$$

Many of the definite integrals may be evaluated easily with the help of the above stated properties, which could have been very difficult otherwise.

The use of these properties in evaluating definite integrals will be illustrated in the following examples.

Example 31.6 Show that

(a) $\int_0^{\frac{\pi}{2}} \log|\tan x|\,dx = 0$ (b) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \pi$

Solution : (a) Let $I = \int_0^{\frac{\pi}{2}} \log|\tan x|\, dx$ ....(i)

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\frac{\pi}{2}} \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx = \int_0^{\frac{\pi}{2}} \log(\cot x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\tan x)^{-1}\,dx = -\int_0^{\frac{\pi}{2}} \log \tan x\,dx$$

$$= -I \qquad \text{[Using (i)]}$$

$\therefore$ $2I = 0$

i.e. $I = 0$ or $\int_0^{\frac{\pi}{2}} \log|\tan x|\,dx = 0$

(b) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$

Let $I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx$ (i)

$\therefore$ $I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx \qquad \text{(ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x}\,dx = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$

or $2I = \pi\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx$

$$= \pi\int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right) dx$$

$$= \pi\left[\tan x - \sec x\right]_0^{\pi}$$

$$= \pi\left[\left(\tan\pi - \sec\pi\right) - \left(\tan 0 - \sec 0\right)\right]$$

$$= \pi\left[0 - (-1) - (0 - 1)\right]$$

$$= 2\pi$$

$\therefore \qquad I = \pi$

Example 31.7 Evaluate

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\,dx$

Solution : (a) Let $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ (i)

Also $$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx$$

(Using the property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$).

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\frac{\pi}{2}} 1.dx$$

$$= \left[x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$\therefore \qquad I = \dfrac{\pi}{4}$

i.e. $$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \frac{\pi}{4}$$

(b) Let $$I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx \qquad \text{(i)}$$

Then $$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x}\, dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \sin x \cos x}\, dx$$

$$= 0$$

$\therefore \qquad I = 0$

Example 31.8 Evaluate (a) $\displaystyle\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}\, dx$ (b) $\displaystyle\int_{-3}^{3} |x+1|\, dx$

Solution : (a) Here $\quad f(x) = \dfrac{xe^{x^2}}{1+x^2} \qquad \therefore \qquad f(-x) = -\dfrac{xe^{x^2}}{1+x^2}$

$$= -f(x)$$

$\therefore$ $f(x)$ is an odd function of x.

$$\therefore \qquad \int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}\, dx = 0$$

(b) $\displaystyle\int_{-3}^{3} |x+1|\,dx$

$$|x+1| = \begin{cases} x+1, \text{ if } x \geq -1 \\ -x-1, \text{ if } x < -1 \end{cases}$$

$$\therefore \quad \int_{-3}^{3} |x+1|\,dx = \int_{-3}^{-1} |x+1|\,dx + \int_{-1}^{3} |x+1|\,dx, \text{ using property (iii)}$$

$$= \int_{-3}^{-1} (-x-1)\,dx + \int_{-1}^{3} (x+1)\,dx$$

$$= \left[\frac{-x^2}{2} - x\right]_{-3}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

$$= -\frac{1}{2} + 1 + \frac{9}{2} - 3 + \frac{9}{2} + 3 - \frac{1}{2} + 1 = 10$$

Example 31.9 Evaluate $\displaystyle\int_{0}^{\frac{\pi}{2}} \log(\sin x)\,dx$

Solution : Let $\quad I = \displaystyle\int_{0}^{\frac{\pi}{2}} \log(\sin x)\,dx \qquad\qquad$ (i)

Also $\qquad I = \displaystyle\int_{0}^{\frac{\pi}{2}} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right]dx, \qquad$ [Using property (iv)]

$$= \int_{0}^{\frac{\pi}{2}} \log(\cos x)\,dx \qquad\qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{0}^{\frac{\pi}{2}} [\log(\sin x) + \log(\cos x)]\,dx = \int_{0}^{\frac{\pi}{2}} \log(\sin x \cos x)\,dx$$

$$= \int_{0}^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx = \int_{0}^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \int_{0}^{\frac{\pi}{2}} \log(2)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2 \qquad \text{(iii)}$$

Again, let $\quad I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx$

Put $2x = t \quad \Rightarrow \quad dx = \frac{1}{2}dt$

When $x = 0$, $t = 0$ and $x = \frac{\pi}{2}, t = \pi$

$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin t)\,dt, \qquad \text{[using property (vi)]}$$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dt \qquad \text{[using property (i)]}$$

$\therefore \quad I_1 = I, \qquad$ [from (i)] $\qquad$ .....(iv)

Putting this value in (iii), we get

$$2I = I - \frac{\pi}{2}\log 2 \quad \Rightarrow \quad I = -\frac{\pi}{2}\log 2$$

Hence, $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = -\frac{\pi}{2}\log 2$

CHECK YOUR PROGRESS 31.2

Evaluate the following integrals :

1. $\int_0^1 xe^{x^2}dx$ $\qquad$ 2. $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\sin x}$ $\qquad$ 3. $\int_0^1 \frac{2x+3}{5x^2+1}dx$

4. $\int_{-5}^{5} |x+2|\,dx$ $\qquad$ 5. $\int_0^2 x\sqrt{2-x}\,dx$ $\qquad$ 6. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}dx$

7. $\int_0^{\frac{\pi}{2}} \log \cos x \, dx$ 8. $\int_{-a}^{a} \frac{x^3 e^{x^4}}{1+x^2} dx$ 9. $\int_0^{\frac{\pi}{2}} \sin 2x \log \tan x \, dx$

10. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x+\cos x} dx$

31.4 APPLICATIONS OF INTEGRATION

Suppose that f and g are two continuous functions on an interval [a, b] such that $f(x) \le g(x)$ for $x \in [a, b]$ that is, the curve y = f (x) does not cross under the curve y = g (x) over [a, b]. Now the question is how to find the area of the region bounded above by y = f (x), below by y = g (x), and on the sides by x = a and x = b.

Again what happens when the upper curve y = f (x) intersects the lower curve y = g (x) at either the left hand boundary x = a , the right hand boundary x = b or both?

31.4.1 Area Bounded by the Curve, x-axis and the Ordinates

Let AB be the curve y = f (x) and CA, DB the two ordinates at x = a and x = b respectively. Suppose y = f (x) is an increasing function of x in the interval $a \le x \le b$.

Let P (x, y) be any point on the curve and $Q(x+\delta x,\ y+\delta y)$ a neighbouring point on it. Draw their ordinates PM and QN.

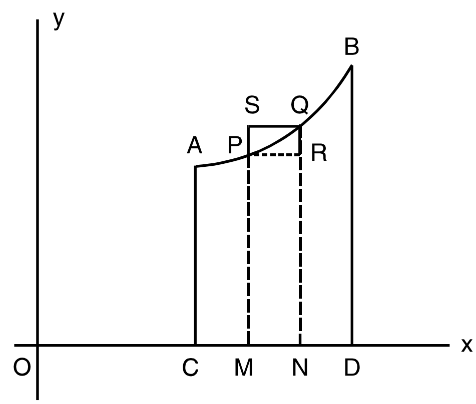


Fig. 31.6

Here we observe that as x changes the area (ACMP) also changes. Let

$$A = \text{Area (ACMP)}$$

Then the area (ACNQ) $= A + \delta A$.

The area (PMNQ)=Area (ACNQ) – Area (ACMP)

$$= A + \delta A - A = \delta A.$$

Complete the rectangle PRQS. Then the area (PMNQ) lies between the areas of rectangles PMNR and SMNQ, that is

δA lies between $y\ \delta x$ and $(y+\delta y)\delta x$

$\Rightarrow \frac{\delta A}{\delta x}$ lies between y and $(y+\delta y)$

In the limiting case when $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$.

$$\therefore \qquad \lim_{\delta x \to 0} \frac{\delta A}{\delta x} \text{ lies between } y \text{ and } \lim_{\delta y \to 0} (y + \delta y)$$

$$\therefore \qquad \frac{dA}{dx} = y$$

Integrating both sides with respect to x, from x = a to x = b, we have

$$\int_a^b y\,dx = \int_a^b \frac{dA}{dx} \cdot dx = [A]_a^b$$

$$= (\text{Area when } x = b) - (\text{Area when } x = a)$$

$$= \text{Area (ACDB)} - 0$$

$$= \text{Area (ACDB)}.$$

Hence $\quad$ Area (ACDB) $= \int_a^b f(x)\,dx$

The area bounded by the curve y = f (x), the x-axis and the ordinates x = a, x = b is

$$\int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where y = f (x) is a continuous single valued function and y does not change sign in the interval $a \le x \le b$.

Example31.10 Find the area bounded by the curve y = x, x-axis and the lines x =0, x = 2.

Solution : The given curve is y = x

$\therefore$ Required area bounded by the curve, x-axis and the ordinates $x = 0,\ x = 2$ (as shown in Fig.31.7)

is $$\int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= 2 - 0 = 2 \text{ square units}$$

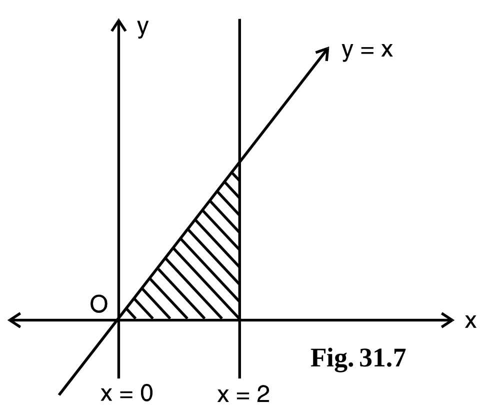


Fig. 31.7

MODULE - VIII
Calculus

Notes

Example 31.11 Find the area enclosed by the circle $x^2 + y^2 = a^2$, and x-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre and radius are (0, 0) and a respectively. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the x-axis and the ordinates x = 0 and x = a.

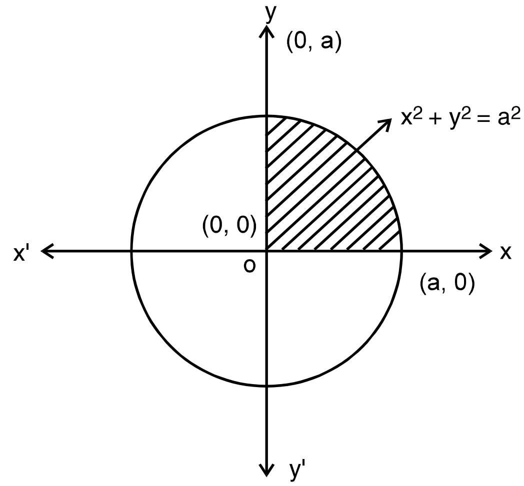


Fig. 31.8

$$\therefore \quad \text{Required area} = \int_0^a y\,dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx,$$

($\because$ y is positive in the first quadrant)

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0$$

$$= \frac{a^2}{2}.\frac{\pi}{2} \quad \left(\because \sin^{-1}1 = \frac{\pi}{2}, \sin^{-1}0 = 0 \right)$$

$$= \frac{\pi a^2}{4} \text{ square units}$$

CHECK YOUR PROGRESS 31.3

1. Find the area bounded by the curve $y = x^2$, x-axis and the lines x = 0, x = 2.
2. Find the area bounded by the curve y = 3 x, x-axis and the lines x = 0 and x = 3.

31.4.2. Area Bounded by the Curve x = f (y) between y-axis and the Lines y = c, y = d

Let AB be the curve x = f (y) and let CA, DB be the abscissae at y = c, y = d respectively.

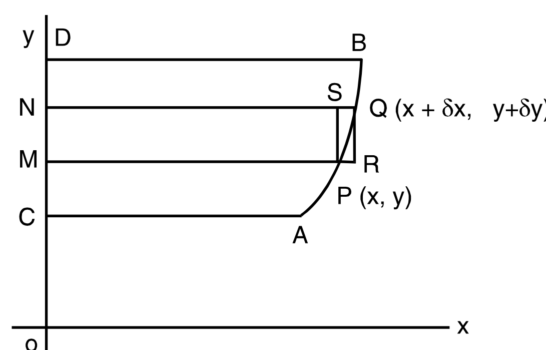


Fig. 31.9

Let P (x, y) be any point on the curve and let $Q(x + \delta x,\ y + \delta y)$ be a neighbouring point on it. Draw PM and QN perpendiculars on y-axis from P and Q respectively. As y changes, the area (ACMP) also changes and hence clearly a function of y. Let A denote the area (ACMP), then the area (ACNQ) will be $A + \delta A$.

The area (PMNQ) = Area (ACNQ) – Area (ACMP) $= A + \delta A - A = \delta A$.

Complete the rectangle PRQS. Then the area (PMNQ) lies between the area (PMNS) and the area (RMNQ), that is,

δA lies between $x\,\delta y$ and $(x+\delta x)\,\delta y$

$$\Rightarrow \quad \frac{\delta A}{\delta y} \text{ lies between } x \text{ and } x+\delta x$$

In the limiting position when $Q \to P$, $\delta x \to 0$ and $\therefore$.

$$\therefore \quad \lim_{\delta y \to 0} \frac{\delta A}{\delta y} \text{ lies between } x \text{ and } \lim_{\delta x \to 0}(x+\delta x)$$

$$\Rightarrow \quad \Rightarrow \frac{dA}{dy} = x$$

Integrating both sides with respect to y, between the limits c to d, we get

$$\int_c^d x\,dy = \int_c^d \frac{dA}{dy}\cdot dy$$

$$= =[A]_c^d$$

$$= (\text{Area when } y=d) - (\text{Area when } y=c)$$

$$= \text{Area (ACDB)} - 0$$

$$= \text{Area (ACDB)}$$

Hence area $\quad$ (ACDB) $= \int_c^d x\,dy = \int_c^d f(y)\,dy$

The area bounded by the curve $x = f(y)$, the y-axis and the lines $y = c$ and $y = d$ is

$$\int_c^d x\,dy \quad \text{or} \quad \int_c^d f(y)\,dy$$

where $x = f(y)$ is a continuous single valued function and x does not change sign in the interval $c \le y \le d$.

Example 31.12 Find the area bounded by the curve $x = y$, y-axis and the lines $y = 0$, $y = 3$.

Solution : The given curve is $x = y$.

$\therefore$ Required area bounded by the curve, y-axis and the lines $y = 0$, $y = 3$ is

$$= \int_0^3 x\,dy$$

$$= \int_0^3 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^3$$

$$= \frac{9}{2} - 0$$

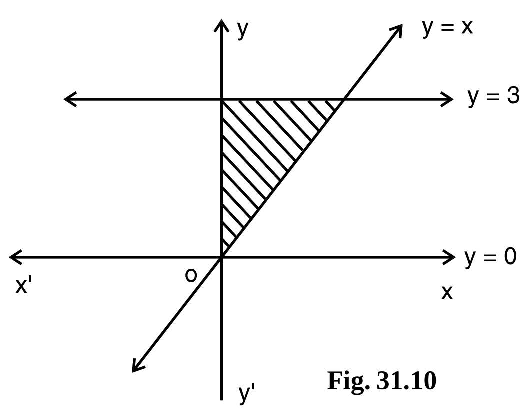


Fig. 31.10

$$= \frac{9}{2} \text{ square units}$$

Example 31.13 Find the area enclosed by the circle $x^2 + y^2 = a^2$ and y-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre is (0, 0) and radius a. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the y-axis and the abscissae y = 0, y = a.

$$\therefore \quad \text{Required area} = \int_0^a x \, dy$$

$$= \int_0^a \sqrt{a^2 - y^2} \, dy$$

(because x is positive in first quadrant)

$$= \left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right) \right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0$$

$$= \frac{\pi a^2}{4} \text{ square units} \qquad \left(\because \sin^{-1}0 = 0, \sin^{-1}1 = \frac{\pi}{2} \right)$$

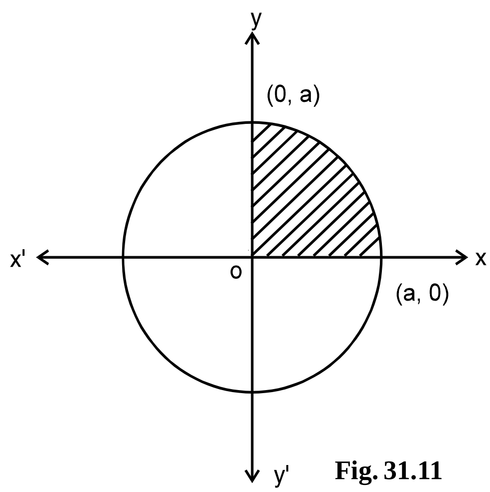


Fig. 31.11

Note : The area is same as in Example 31.11, the reason is the given curve is symmetrical about both the axes. In such problems if we have been asked to find the area of the curve, without any restriction we can do by either method.

Example 31.14 Find the whole area bounded by the circle $x^2 + y^2 = a^2$.

Solution : The equation of the curve is $x^2 + y^2 = a^2$.

The circle is symmetrical about both the axes, so the whole area of the circle is four times the area os the circle in the first quadrant, that is,

Area of circle = 4 × area of OAB

$$= 4 \times \frac{\pi a^2}{4} \text{(From Example 12.11 and 12.13)} = \pi a^2$$

square units

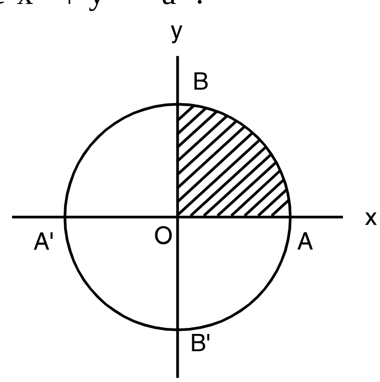


Fig. 31.12

Example 31.15 Find the whole area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Solution : The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The ellipse is symmetrical about both the axes and so the whole area of the ellipse is four times the area in the first quadrant, that is,
Whole area of the ellipse $= 4 \times$ area (OAB)

In the first quadrant,

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2} \quad \text{or} \quad y=\frac{b}{a}\sqrt{a^2-x^2}$$

Now for the area (OAB), x varies from 0 to a

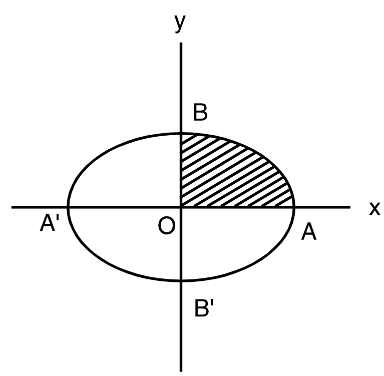


Fig. 31.13

$$\therefore \qquad \text{Area (OAB)} = \int_0^a y\,dx$$

$$=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=\frac{b}{a}\left[0+\frac{a^2}{2}\sin^{-1}1-0-\frac{a^2}{2}\sin^{-1}0\right]$$

$$=\frac{ab\pi}{4}$$

Hence the whole area of the ellipse

$$=4\times\frac{ab\pi}{4}$$

$$=\pi ab. \text{ square units}$$

31.4.3 Area between two Curves

Suppose that f (x) and g (x) are two continuous and non-negative functions on an interval [a, b] such that $f(x) \geq g(x)$ for all $x \in [a,\ b]$ that is, the curve y = f (x) does not cross under the curve y = g (x) for $x \in [a, b]$. We want to find the area bounded above by y = f (x), below by y = g (x), and on the sides by x = a and x = b.

Let A= [Area under y = f (x)] – [Area under y = g (x)] .....(1)

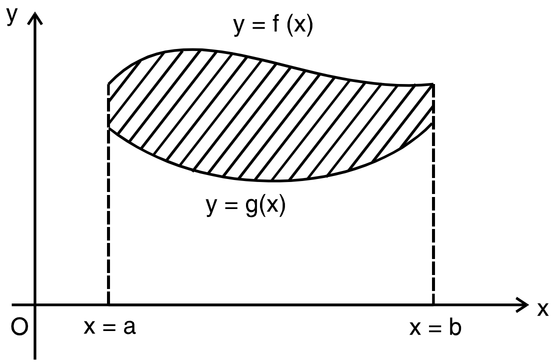


Fig. 31.14

Now using the definition for the area bounded by the curve $y = f(x)$, x-axis and the ordinates x = a and x = b, we have

Area under

$$y = f(x) = \int_a^b f(x)\,dx \qquad .....(2)$$

Similarly,Area under $y = g(x) = \int_a^b g(x)\,dx$ .....(3)

Using equations (2) and (3) in (1), we get

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx \qquad .....(4)$$

What happens when the function g has negative values also? This formula can be extended by translating the curves f (x) and g (x) upwards until both are above the x-axis. To do this let-m be the minimum value of g (x) on [a, b] (see Fig. 31.15).

Since $\quad g(x) \geq -m \quad \Rightarrow \quad g(x) + m \geq 0$

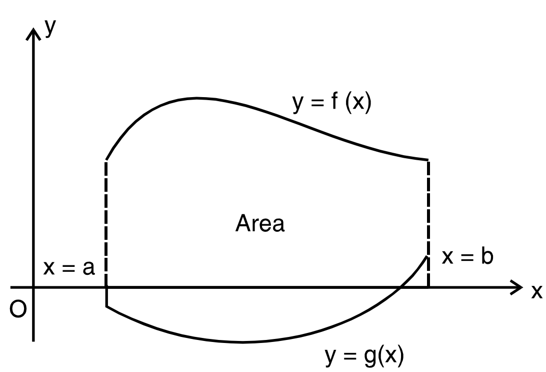


Fig. 31.15

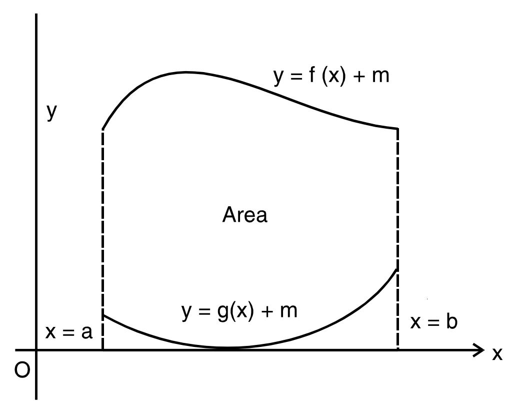


Fig. 31.16

Now, the functions $g(x) + m$ and $f(x) + m$ are non-negative on [a, b] (see Fig. 31.16). It is intuitively clear that the area of a region is unchanged by translation, so the area A between f and g is the same as the area between $g(x) + m$ and $f(x) + m$. Thus,

$A = [\text{ area under } y = [f(x) + m]] - [\text{area under } y = [g(x) + m]]$ .....(5)

Now using the definitions for the area bounded by the curve y = f (x), x-axis and the ordinates x = a and x = b, we have

$$\text{Area under } y = f(x) + m = \int_a^b [f(x) + m]dx \qquad .....(6)$$

and

$$\text{Area under } y = g(x) + m = \int_a^b [g(x) + m]dx \qquad (7)$$

The equations (6), (7) and (5) give

$$A = \int_a^b [f(x) + m]dx - \int_a^b [g(x) + m]dx$$

$$= \int_a^b [f(x) - g(x)]dx$$

which is same as (4) Thus,

If f (x) and g (x) are continuous functions on the interval [a, b], and $f(x) \geq g(x), \forall x \in [a, b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$= \int_a^b [f(x) - g(x)]dx$$

$$= \frac{34}{3} \text{ square units}$$

If the curves intersect then the sides of the region where the upper and lower curves intersect reduces to a point, rather than a vertical line segment.

Example 31.16 Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution : We know that $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and vertex is origin and y = x + 6 is the equation of the straight line. (See Fig. 31.17).

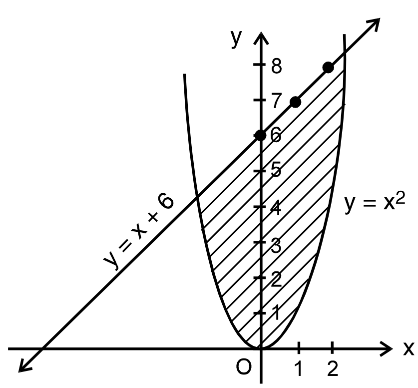


Fig. 31.17

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is y = x +6. These two curves intersect at two points, say A and B. Solving these two equations we get

$$x^2 = x + 6 \quad \Rightarrow \quad x^2 - x - 6 = 0$$

$$\Rightarrow \quad (x-3)(x+2) = 0 \quad \Rightarrow \quad x = 3, -2$$

When x = 3, y = 9 and when x = –2, y = 4

$$\therefore \text{ The required area } = \int_{-2}^{3}\left[(x+6) - x^2\right]dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3}\right)$$

$$= \frac{125}{6} \text{ square units}$$

Example 31.17 Find the area bounded by the curves $y^2 = 4x$ and y = x.

Solution : We know that $y^2 = 4x$ the equation of the parabola which is symmetric about the x-axis and origin is the vertex. y = x is the equation of the straight line (see Fig. 31.18).

A sketch of the region shows that the lower boundary is y = x and the upper boundary is $y^2 = 4x$. These two curves intersect at two points O and A. Solving these two equations, we get

$$\frac{y^2}{4} - y = 0$$

$$\Rightarrow \quad y(y-4) = 0$$

$$\Rightarrow \quad y = 0, 4$$

When y = 0, x = 0 and when y = 4, x = 4.

Here $f(x) = (4x)^{\frac{1}{2}}, g(x) = x, a = 0, b = 4$

Therefore, the required area is

$$= \int_0^4 \left(2x^{\frac{1}{2}} - x\right)dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^4$$

$$= \frac{32}{3} - 8$$

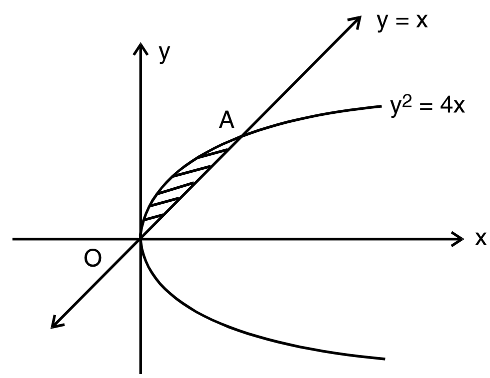


Fig. 31.18

$$= \frac{8}{3} \quad \text{square units}$$

Example 31.18 Find the area common to two parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Solution : We know that $y^2 = 4ax$ and $x^2 = 4ay$ are the equations of the parabolas, which are symmetric about the x-axis and y-axis respectively.

Also both the parabolas have their vertices at the origin (see Fig. 31.19).

A sketch of the region shows that the lower boundary is $x^2 = 4ay$ and the upper boundary is $y^2 = 4ax$. These two curves intersect at two points O and A. Solving these two equations, we have

$$\frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow \qquad x\left(x^3 - 64a^3\right) = 0$$

$$\Rightarrow \qquad x = 0, 4a$$

Hence the two parabolas intersect at point (0, 0) and (4a, 4a).

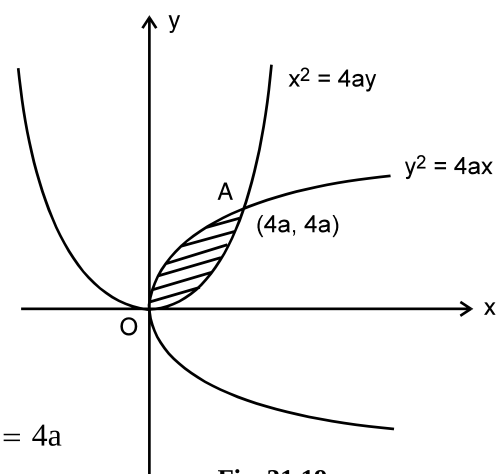


Fig. 31.19

Here $\quad f(x) = \sqrt{4ax}, g(x) = \frac{x^2}{4a}, a = 0 \text{ and } b = 4a$

Therefore, required area

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] dx$$

$$= \left[\frac{2.2\sqrt{a}\,x^{\frac{3}{2}}}{3} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16}{3} a^2 \text{ square units}$$

CHECK YOUR PROGRESS 31.4

1. Find the area of the circle $x^2 + y^2 = 9$

2. Find the area of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

3. Find the area of the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

4. Find the area bounded by the curves $y^2 = 4ax$ and $y = \dfrac{x^2}{4a}$

5. Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.

6. Find the area enclosed by the curves $y = x^2$ and $y = x + 2$

LET US SUM UP

- If f is continuous in [a, b] and F is an anti derivative of f in [a, b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- If f and g are continuous in [a, b] and c is a constant, then

(i) $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

(ii) $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(iii) $$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

- The area bounded by the curve y = f (x), the x-axis and the ordinates

$$x = a, x = b \text{ is } \int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where $y = f(x)$ is a continuous single valued function and y does not change sign in the interval $a \le x \le b$

- If f (x) and g (x) are continuous functions on the interval [a, b] and $f(x) \geq g(x)$, for all $x \in [a, b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$\int_a^b [f(x) - g(x)]\,dx$$

SUPPORTIVE WEB SITES

http://mathworld.wolfram.com/DefiniteIntegral.html

http://www.mathsisfun.com/calculus/integration-definite.html

TERMINAL EXERCISE

Evaluate the following integrals (1 to 5) as the limit of sum.

1. $\int_a^b x\,dx$ 2. $\int_a^b x^2\,dx$ 3. $\int_0^2 \left(x^2 + 1\right) dx$

Evaluate the following integrals (4 to 20)

4. $\int_0^2 \sqrt{a^2 - x^2}\,dx$ 5. $\int_0^{\frac{\pi}{2}} \sin 2x\,dx$ 6. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$

7. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ 8. $\int_0^1 \sin^{-1} x\,dx$ 9. $\int_0^1 \frac{1}{\sqrt{1 - x^2}}\,dx$

10. $\int_3^4 \frac{1}{x^2 - 4}\,dx$ 11. $\int_0^{\pi} \frac{1}{5 + 3\cos\theta}\,d\theta$ 12. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx$

13. $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$ 14. $\int_0^2 x\sqrt{x + 2}\,dx$ 15. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\,\cos^5\theta\,d\theta$

16. $\int_0^{\pi} x \log \sin x\,dx$ 17. $\int_0^{\pi} \log(1 + \cos x)\,dx$ 18. $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx$

19. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$

20. $\int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$

21. Find the area bounded by the curve $x = y^2$, y – axis and lines y=0, y=2.

22. Find the area of the region bounded by the curve $y = x^2$ and y=x.

23. Find the area bounded by the curve $y^2 = 4x$ and straight line x=3.

24. Find the area of triangular region whose vertices have coordinates (1,0), (2,2) and (3.1)

25. Find the area of the smaller region bounded by the cllipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the straight line $\frac{x}{3} + \frac{y}{2} = 1$

26. Find the area of the region bounded by the paralal $y = x^2$ and the curve $y = |x|$

ANSWERS

CHECK YOUR PROGRESS 31.1

1. $\frac{35}{2}$ 2. $e - \frac{1}{e}$

3. (a) $\frac{\sqrt{2}-1}{\sqrt{2}}$ (b) 2 (c) $\frac{\pi}{4}$ (d) $\frac{64}{3}$

CHECK YOUR PROGRESS 31.2

1. $\frac{e-1}{2}$ 2. $\frac{2}{3}\tan^{-1}\frac{1}{3}$ 3. $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

4. 29 5. $\frac{24\sqrt{2}}{15}$ 6. $\frac{\pi}{4}$

7. $-\frac{\pi}{2}\log 2$ 8. 0 9. 0 10. $\frac{1}{2}\left[\frac{\pi}{2} - \log 2\right]$

CHECK YOUR PROGRESS 31.3

1. $\frac{8}{3}$ sq. units 2. $\frac{27}{2}$ sq. units

CHECK YOUR PROGRESS 31.4

1. 9π sq. units 2. 6π sq. units 3. 20π sq. units

4. $\frac{16}{3}a^2$ sq. units 5. $\frac{16}{3}$ sq. units 6. $\frac{9}{2}$ sq. units

TERMINAL EXERCISE

1. $\frac{b^2 - a^2}{2}$ 2. $\frac{b^3 - a^3}{3}$ 3. $\frac{14}{3}$ 4. $\frac{\pi a^2}{4}$

5. 1 6. $\frac{1}{2}\log 2$ 7. $\frac{\pi}{4}$ 8. $\frac{\pi}{2} - 1$

9. $\frac{\pi}{2}$ 10. $\frac{1}{4}\log\frac{5}{3}$ 11. $\frac{\pi}{4}$ 12. $1 - \log 2$

13. $\frac{2}{3}$

14. $\frac{16}{15}\left(2+\sqrt{2}\right)$

15. $\frac{64}{231}$

16. $-\frac{\pi^2}{2}\log 2$

17. $-\pi \log 2$

18. $\frac{\pi^2}{4}$

19. $\frac{1}{\sqrt{2}}\log\left(1+\sqrt{2}\right)$

20. $\frac{\pi}{8}\log 2$

21. $\frac{8}{3}$ square unit.

22. $\frac{1}{6}$ Square unit

23. $8\sqrt{3}$ Square unit

24. $\frac{3}{2}$ Square unit

25. $\frac{3}{2}(\pi-2)$ Square unit

26. $\frac{1}{3}$ Square unit

32

DIFFERENTIAL EQUATIONS

Having studied the concept of differentiation and integration, we are now faced with the question where do they find an application.

In fact these are the tools which help us to determine the exact takeoff speed, angle of launch, amount of thrust to be provided and other related technicalities in space launches.

Not only this but also in some problems in Physics and Bio-Sciences, we come across relations which involve derivatives.

One such relation could be $\frac{ds}{dt} = 4.9\ t^2$ where *s* is distance and *t* is time. Therefore, $\frac{ds}{dt}$ represents velocity (rate of change of distance) at time *t*.

Equations which involve derivatives as their terms are called differential equations. In this lesson, we are going to learn how to find the solutions and applications of such equations.

OBJECTIVES

After studying this lesson, you will be able to :

- define a differential equation, its order and degree;
- determine the order and degree of a differential equation;
- form differential equation from a given situation;
- illustrate the terms "general solution" and "particular solution" of a differential equation through examples;
- solve differential equations of the following types :

  (i) $\frac{dy}{dx} = f(x)$ (ii) $\frac{dy}{dx} = f(x)\,g(y)$

  (iii) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (iv) $\frac{dy}{dx} + P(x)\,y = Q(x)$

- find the particular solution of a given differential equation for given conditions.

EXPECTED BACKGROUND KNOWLEDGE

- Integration of algebraic functions, rational functions and trigonometric functions

32.1 DIFFERENTIAL EQUATIONS

As stated in the introduction, many important problems in Physics, Biology and Social Sciences, when formulated in mathematical terms, lead to equations that involve derivatives. Equations which involve one or more differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ (or differentials) etc. and independent and dependent variables are called differential equations.

For example,

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + y = 0$ (iii) $xdx + ydy = 0$

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ (vi) $y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

32.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Order : It is the order of the highest derivative occurring in the differential equation.

Degree : It is the degree of the highest order derivative in the differential equation.

	Differential Equation	Order	Degree
(i)	$\frac{dy}{dx} = \sin x$	One	One
(ii)	$\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x$	One	Two
(iii)	$\left(\frac{d^2s}{dt^2}\right)^2 + t^2\left(\frac{ds}{dt}\right)^4 = 0$	Two	Two
(iv)	$\frac{d^3v}{dr^3} + \frac{2}{r}\frac{dv}{dr} = 0$	Three	One
(v)	$\left(\frac{d^4y}{dx^4}\right)^2 + x^3\left(\frac{d^3y}{dx^3}\right)^5 = \sin x$	Four	Two

Example 32.1 Find the order and degree of the differential equation :

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 0$$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 0 \qquad \text{or} \qquad \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$$

Hence order of the diferential equation is 2 and the degree of the differential equation is 1.

Note : Degree of a differential equation is defiend if it is a palynomial equation in terms of its derivatives.

32.3 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variable and all of its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**. A differential equation which is not linear is called non-linear differential equation . For example, the differential equations

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{and} \quad \cos^2 x \frac{d^3y}{dx^3} + x^3 \frac{dy}{dx} + y = 0 \text{ are linear.}$$

The differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{y}{x} = \log x$ is non-linear as degree of $\frac{dy}{dx}$ is two.

Further the differential equation $y \frac{d^2y}{dx^2} - 4 = x$ is non-linear because the dependent variable *y* and its derivative $\frac{d^2y}{dx^2}$ are multiplied together.

32.4 FORMATION OF A DIFFERENTIAL EQUATION

Consider the family of all straight lines passing through the origin (see Fig. 28.1).

This family of lines can be represented by

$$y = mx \qquad .....(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = m \qquad .....(2)$$

From (1) and (2), we get

$$y = x\frac{dy}{dx} \qquad .....(3)$$

So $y = mx$ and $y = x\frac{dy}{dx}$ represent the same family.

Clearly equation (3) is a differential equation.

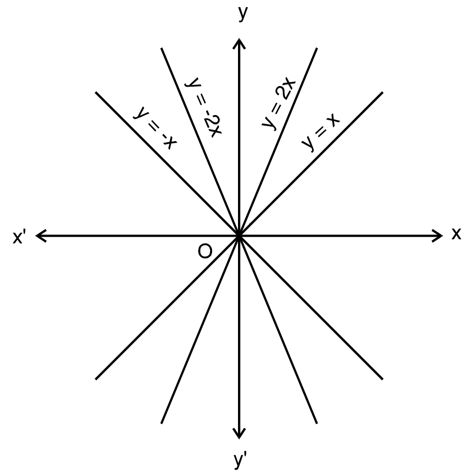


Fig. 32.1

Working Rule : To form the differential equation corresponding to an equation involving two variables, say *x* and *y* and some arbitrary constants, say, *a*, *b*, *c*, etc.

(i) Differentiate the equation as many times as the number of arbitrary constants in the equation.

(ii) Eliminate the arbitrary constants from these equations.

Remark

If an equation contains n arbitrary constants then we will obtain a differential equation of n^{th} order.

Example 32.2 Form the differential equation representing the family of curves.

$$y = ax^2 + bx. \qquad ......(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = 2ax + b \qquad ......(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = 2a \qquad ......(3)$$

$$\Rightarrow \qquad a = \frac{1}{2}\frac{d^2y}{dx^2} \qquad ......(4)$$

(The equation (1) contains two arbitrary constants. Therefore, we differentiate this equation two times and eliminate 'a' and 'b').

On putting the value of 'a' in equation (2), we get

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + b$$

$$\Rightarrow \qquad b = \frac{dy}{dx} - x\frac{d^2y}{dx^2} \qquad ......(5)$$

Substituting the values of 'a' and 'b' given in (4) and (5) above in equation (1), we get

$$y = x^2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

or $$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

or $$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$$

or $$\frac{x^2}{2}\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

which is the required differential equation.

Example 32.3 Form the differential equation representing the family of curves

$$y = a\cos(x + b).$$

Solution : $$y = a\cos(x + b) \qquad .....(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = -a\sin(x + b) \qquad ....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -a\cos(x + b) \qquad .....(3)$$

From (1) and (3), we get

$$\frac{d^2y}{dx^2} = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

Example 32.4 Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Solution : As the centre lies on the x-axis, its coordinates will be (a, 0).

Since each circle passes through the origin, its radius is a.

Then the equation of any circle will be

$$(x - a)^2 + y^2 = a^2 \qquad (1)$$

To find the corresponding differential equation, we differentiate equation (1) and get

$$2(x-a)+2y\frac{dy}{dx}=0$$

or $$x-a+y\frac{dy}{dx}=0$$

or $$a=y\frac{dy}{dx}+x$$

Substituting the value of 'a' in equation (1), we get

$$\left(x-y\frac{dy}{dx}-x\right)^2+y^2=\left(y\frac{dy}{dx}+x\right)^2$$

or $$\left(y\frac{dy}{dx}\right)^2+y^2=x^2+\left(y\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}$$

or $$y^2=x^2+2xy\frac{dy}{dx}$$

which is the required differential equation.

Remark

If an equation contains one arbitrary constant then the corresponding differential equation is of the first order and if an equation contains two arbitrary constants then the corresponding differential equation is of the second order and so on.

or $$\frac{dr}{dt}=k$$

which is the required differential equation.

CHECK YOUR PROGRESS 32.1

1. Find the order and degree of the differential equation

$$y=x\frac{dy}{dx}+1$$

2. Write the order and degree of each of the following differential equations.

(a) $$\left(\frac{ds}{dt}\right)^4+3s\frac{d^2s}{dt^2}=0$$

(b) $$\left(\frac{d^2s}{dt^2}\right)^2+3\left(\frac{ds}{dt}\right)^3+4=0$$

3. State whether the following differential equations are linear or non-linear.

(a) $\left(xy^2 - x\right)dx + \left(y - x^2y\right)dy = 0$ (b) $dx + dy = 0$

(c) $\frac{dy}{dx} = \cos x$ (d) $\frac{dy}{dx} + \sin^2 y = 0$

4. Form the differential equation corresponding to

$(x - a)^2 + (y - b)^2 = r^2$ by eliminating 'a' and 'b'.

5. (a) Form the differential equation corresponding to

$y^2 = m\left(a^2 - x^2\right)$

(b) Form the differential equation corresponding to

$y^2 - 2ay + x^2 = a^2$, where a is an arbitrary constant.

(c) Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-3x}$ where A and B are arbitrary constants.

(d) Find the differential equation of all straight lines passing through the point (3,2).

(e) Find the differential equation of all the circles which pass through origin and whose centres lie on y-axis.

32.5 GENERAL AND PARTICULAR SOLUTIONS

Finding solution of a differential equation is a reverse process. Here we try to find an equation which gives rise to the given differential equation through the process of differentiations and elimination of constants. The equation so found is called the primitive or the solution of the differential equation.

Remarks

(1) If we differentiate the primitive, we get the differential equation and if we integrate the differential equation, we get the primitive.

(2) Solution of a differential equation is one which satisfies the differential equation.

Example 32.5 Show that $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are arbitrary constants, is a solution of the differential equation :

$$\frac{d^2y}{dx^2} + y = 0$$

Solution : We are given that

$$y = C_1 \sin x + C_2 \cos x \qquad .....(1)$$

Differentiating both sides of (1), we get

$$\frac{dy}{dx} = C_1 \cos x - C_2 \sin x \qquad .....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -C_1 \sin x - C_2 \cos x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

$$\frac{d^2y}{dx^2} + y = C_1 \sin x + C_2 \cos x + (-C_1 \sin x - C_2 \cos x)$$

or
$$\frac{d^2y}{dx^2} + y = 0$$

In integration, the arbitrary constants play important role. For different values of the constants we get the different solutions of the differential equation.

A solution which contains as many as arbitrary constants as the order of the differential equation is called the **General Solution** or complete primitive.

If we give the particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a **Particular Solution**.

Remark

General Solution contains as many arbitrary constants as is the order of the differential equation.

Example 32.6 Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation.

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}$$

Solution : We have $y = cx + \frac{a}{c}$ .....(1)

Differentiating (1), we get

$$\frac{dy}{dx} = c \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{c}$$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation, we have

$$x(c) + a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$$

$\Rightarrow$ R.H.S. = L.H.S.

Hence $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

Example 32.7 If $y = 3x^2 + C$ is the general solution of the differential equation $\frac{dy}{dx} - 6x = 0$, then find the particular solution when y = 3, x = 2.

Solution : The general solution of the given differential equation is given as

$$y = 3x^2 + C \qquad ....(1)$$

Now on substituting y = 3, x = 2 in the above equation , we get

$$3 = 12 + C \qquad \text{or} \qquad C = -9$$

By substituting the value of C in the general solution (1), we get

$$y = 3x^2 - 9$$

which is the required particular solution.

32.6 TECHNIQUES OF SOLVING IN G. A DIFFERENTIAL EQUATION

32.6.1 When Variables are Separable

(i) Differential equation of the type $\frac{dy}{dx} = f(x)$

Consider the differential equation of the type $\frac{dy}{dx} = f(x)$

or $\qquad dy = f(x)\, dx$

On integrating both sides, we get

$$\int dy = \int f(x)\, dx$$

$$y = \int f(x)\, dx + c$$

where c is an arbitrary constant. This is the general solution.

Note : It is necessary to write c in the general solution, otherwise it will become a particular solution.

Example 32.8 Solve

$$(x + 2)\frac{dy}{dx} = x^2 + 4x - 5$$

Solution : The given differential equation is $(x + 2)\frac{dy}{dx} = x^2 + 4x - 5$

or $\dfrac{dy}{dx} = \dfrac{x^2 + 4x - 5}{x + 2}$ or $\dfrac{dy}{dx} = \dfrac{x^2 + 4x + 4 - 4 - 5}{x + 2}$

or $\dfrac{dy}{dx} = \dfrac{(x + 2)^2}{x + 2} - \dfrac{9}{x + 2}$ or $\dfrac{dy}{dx} = x + 2 - \dfrac{9}{x + 2}$

or $$dy = \left(x + 2 - \frac{9}{x + 2} \right) dx \qquad .....(1)$$

On integrating both sides of (1), we have

$$\int dy = \int \left(x + 2 - \frac{9}{x + 2} \right) dx \quad \text{or} \quad y = \frac{x^2}{2} + 2x - 9 \log|x + 2| + c,$$

where c is an arbitrary constant, is the required general solution.

Example 32.9 Solve

$$\frac{dy}{dx} = 2x^3 - x$$

given that y = 1 when x = 0

Solution : The given differential equation is $\dfrac{dy}{dx} = 2x^3 - x$

or $$dy = \left(2x^3 - x \right) dx \qquad ...(1)$$

On integrating both sides of (1), we get

$$\int dy = \int \left(2x^3 - x \right) dx \quad \text{or} \quad y = 2.\frac{x^4}{4} - \frac{x^2}{2} + C$$

or $$y = \frac{x^4}{2} - \frac{x^2}{2} + C \qquad .....(2)$$

where C is an arbitrary constant.

Since y = 1 when x = 0, therefore, if we substitute these values in (2) we will get

$$1 = 0 - 0 + C \quad \Rightarrow \quad C = 1$$

Now, on putting the value of C in (2), we get

$$y = \frac{1}{2}\left(x^4 - x^2 \right) + 1 \quad \text{or} \quad y = \frac{1}{2} x^2 \left(x^2 - 1 \right) + 1$$

which is the required particular solution.

(ii) Differential equations of the type $\dfrac{dy}{dx} = f(x) \cdot g(y)$

Consider the differential equation of the type

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

or $$\frac{dy}{g(y)} = f\ (x)\ dx \qquad .....(1)$$

In equation (1), x's and y's have been separated from one another. Therefore, this equation is also known differential equation with variables separable.

To solve such differential equations, we integrate both sides and add an arbitrary constant on one side.

To illustrate this method, let us take few examples.

Example 32.10 Solve

$$\left(1+x^2\right)dy = \left(1+y^2\right)dx$$

Solution : The given differential equation

$$\left(1+x^2\right)dy = \left(1+y^2\right)dx$$

can be written as $$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$ (Here variables have been seperated)

On integrating both sides of (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or $$\tan^{-1} y = \tan^{-1} x + C$$

where C is an arbitrary constant.

This is the required solution.

Example 32.11 Find the particular solution of

$$\frac{dy}{dx} = \frac{2x}{3y^2+1}$$

when y (0) = 3 (i.e. when x = 0, y = 3).

Solution : The given differential equation is

$$\frac{dy}{dx} = \frac{2x}{3y^2+1} \text{ or } \left(3y^2+1\right)dy = 2x\ dx \text{ (Variables separated)} \qquad ....(1)$$

If we integrate both sides of (1), we get

$$\int \left(3y^2+1\right)dy = \int 2x\ dx,$$

where C is an arbitrary constant.

$$y^3 + y = x^2 + C \qquad ....(2)$$

It is given that, $y(0) = 3$.

$\therefore$ on substituting $y = 3$ and $x = 0$ in (2), we get

$$27 + 3 = C$$

$$\therefore \qquad C = 30$$

Thus, the required particular solution is

$$y^3 + y = x^2 + 30$$

32.6.2 Homogeneous Differential Equations

Consider the following differential equations :

(i) $\quad y^2 + x^2 \dfrac{dy}{dx} = xy \dfrac{dy}{dx}$ $\qquad$ (ii) $\quad \left(x^3 + y^3\right) dx - 3xy^2 dy = 0$

(iii) $\quad \dfrac{dy}{dx} = \dfrac{x^3 + xy^2}{y^2 x}$

In equation (i) above, we see that each term except $\dfrac{dy}{dx}$ is of degree 2

[as degree of y^2 is 2, degree of x^2 is 2 and degree of xy is 1 + 1 =2]

In equation (ii) each term except $\dfrac{dy}{dx}$ is of degree 3.

In equation (iii) each term except $\dfrac{dy}{dx}$ is of degree 3, as it can be rewritten as

$$y^2 x \frac{dy}{dx} = x^3 + x y^2$$

Such equations are called **homogeneous equations**.

Remarks

Homogeneous equations do not have constant terms.

For example, differential equation

$$\left(x^2 + 3yx\right) dx - \left(x^3 + x\right) dy = 0$$

is not a homogeneous equation as the degree of the function except $\dfrac{dy}{dx}$ in each term is not the same. [degree of x^2 is 2, that of 3yx is 2, of x^3 is 3, and of x is 1]

32.6.3 Solution of Homogeneous Differential Equation :

To solve such equations, we proceed in the following manner :

(i) write one variable = v. (the other variable).
(i.e. either y = vx or x = vy)

(ii) reduce the equation to separable form

(iii) solve the equation as we had done earlier.

Example 32.12 Solve

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

Solution : The given differential equation is

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

or $$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} \qquad .....(1)$$

It is a homogeneous equation of degree two. (Why?)

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (1), we have

$$v + x\frac{dv}{dx} = \frac{x^2 + 3x.vx + (vx)^2}{x^2} \qquad \text{or} \qquad v + x\frac{dv}{dx} = x^2\left[\frac{1 + 3v + v^2}{x^2}\right]$$

or $$v + x\frac{dv}{dx} = 1 + 3v + v^2 \qquad \text{or} \qquad x\frac{dv}{dx} = 1 + 3v + v^2 - v$$

or $$x\frac{dv}{dx} = v^2 + 2v + 1 \qquad \text{or} \qquad \frac{dv}{v^2 + 2v + 1} = \frac{dx}{x}$$

or $$\frac{dv}{(v+1)^2} = \frac{dx}{x} \qquad .....(2)$$

Further on integrating both sides of (2), we get

$$\frac{-1}{v+1} + C = \log|x|,$$ where C is an arbitrary constant.

On substituting the value of v, we get

$$\frac{x}{y+x} + \log|x| = C$$ which is the required solution.

Note: If the Homoqeneous differential equation is written in the form $\frac{dx}{qy} = \frac{P(x,y)}{Q(x,y)}$ then x=vy is substituted to find solution.

32.6.4 Differential Equation of the type $\frac{dy}{qx} + py = Q$,where P and Q are functions of x only.

Consider the equation

$$\frac{dy}{dx} + Py = Q \qquad .....(1)$$

where P and Q are functions of x. This is linear equation of order one.

To solve equation (1), we first multiply both sides of equation (1) by $e^{\int Pdx}$ (called integrating factor) and get

$$e^{\int Pdx}\frac{dy}{dx} + Py\, e^{\int Pdx} = Qe^{\int Pdx}$$

or $$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx} \qquad .....(2)$$

$$\left[\because \frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}\frac{dy}{dx} + Py.e^{\int Pdx}\right]$$

On integrating, we get

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C \qquad .....(3)$$

where C is an arbitrary constant,

or $$y = e^{-\int Pdx}\left[\int Qe^{\int Pdx}dx + C\right]$$

Note : $e^{\int Pdx}$ is called the integrating factor of the equation and is written as I.F in short.

Remarks

(i) We observe that the left hand side of the linear differential equation (1) has become $\frac{d}{dx}\left(ye^{\int Pdx}\right)$ after the equation has been multiplied by the factor $e^{\int Pdx}$.

(ii) The solution of the linear differential equation

$$\frac{dy}{dx} + Py = Q$$

P and Q being functions of x only is given by

$$ye^{\int Pdx} = \int Q\left(e^{\int Pdx}\right)dx + C$$

(iii) The coefficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing the equation by it throughout.

(iv) Some differential equations become linear differential equations if y is treated as the independent variable and x is treated as the dependent variable.

For example, $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only, is also a linear differential equation of the first order.

In this case $\quad$ $I.F. = e^{\int P dy}$

and the solution is given by

$$x\,(I.F.) = \int Q.(I.F.)dy + C$$

Example 32.13 Solve

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

Solution : Here $P = \frac{1}{x}$, $Q = e^{-x}$ (Note that both P an Q are functions of x)

I.F. (Integrating Factor) $e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x \qquad (x > 0)$

$\therefore$ Solution of the given equation is:

$$yx = \int x e^{-x} dx + C$$

where C is an arbitrary constant

or $\quad xy = -xe^{-x} + \int e^{-x} dx + C$

or $\quad xy = -xe^{-x} - e^{-x} + C$

or $\quad xy = -e^{-x}(x+1) + C$

or $\quad y = -\left(\frac{x+1}{x}\right)e^{-x} + \frac{C}{x}$

Note: In the solution x > 0.

Example 32.14 Solve :

$$\sin x \frac{dy}{dx} + y \cos x = 2\sin^2 x \cos x$$

Solution : The given differential equation is

$$\sin x \frac{dy}{dx} + y \cos x = 2\sin^2 x \cos x$$

or $\quad \frac{dy}{dx} + y \cot x = 2 \sin x \cos x \qquad .....(1)$

Here $\quad P = \cot x, Q = 2 \sin x \cos x$

$$I.F. = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

$\therefore$ Solution of the given equation is:

$$y \sin x = \int 2\sin^2 x \cos x \, dx + C$$

where C is an arbitrary constant (sin x >0)

or $$y \sin x = \frac{2}{3}\sin^3 x + C,$$ which is the required solution.

Example 32.15 Solve $$\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

Solution : The given differential equation is

$$\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

or $$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

or $$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \qquad ....(1)$$

which is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y only.

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ Solution of the given equation is:

$$x - e^{\tan -1 y} = \left(\frac{e^{\tan -1 y}}{1+y^2}\right) e^{\tan -1 y} dy + c.$$

where C is an arbitrary constant let $t = \tan^{-1} y$ therefore $dt = \frac{1}{1+y^2} dy$

or $$\left(e^{\tan^{-1} y}\right) x = \int e^t . tdt + C,$$

or $$\left(e^{\tan^{-1} y}\right) x = te^t - \int e^t + C$$

or $$\left(e^{\tan^{-1} y}\right) x = te^t - e^t + C$$

or $$\left(e^{\tan^{-1} y}\right) x = \tan^{-1} y \, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C \qquad (\text{on putting } t = \tan^{-1} y)$$

or $$x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

CHECK YOUR PROGRESS 32.2

1. (i) Is $y = \sin x$, a solution of $\frac{d^2y}{dx^2} + y = 0$?

   (ii) Is $y = x^3$, a solution of $x\frac{dy}{dx} - 4y = 0$?

2. Given below are some solutions of the differential equation $\frac{dy}{dx} = 3x$.

   State which are particular solutions and which are general solutions.

   (i) $2y = 3x^2$ (ii) $y = \frac{3}{2}x^2 + 2$

   (iii) $2y = 3x^2 + C$ (iv) $y = \frac{3}{2}x^2 + 3$

3. State whether the following differential equations are homogeneous or not ?

   (i) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ (ii) $\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0$

   (iii) $(x+2)\frac{dy}{dx} = x^2 + 4x - 9$ (iv) $\left(x^3 - yx^2\right)dy + \left(y^3 + x^3\right)dx = 0$

4. (a) Show that $y = a\sin 2x$ is a solution of $\frac{d^2y}{dx^2} + 4y = 0$

   (b) Verify that $y = x^3 + ax^2 + c$ is a solution of $\frac{d^3y}{dx^3} = 6$

5. The general solution of the differential equation

   $\frac{dy}{dx} = \sec^2 x$ is $y = \tan x + C$.

   Find the particular solution when

   (a) $x = \frac{\pi}{4}, y = 1$ (b) $x = \frac{2\pi}{3}, y = 0$

6. Solve the following differential equations :

   (a) $\frac{dy}{dx} = x^5 \tan^{-1}\left(x^3\right)$ (b) $\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$

   (c) $\left(1 + x^2\right)\frac{dy}{dx} = x$ (d) $\frac{dy}{dx} = x^2 + \sin 3x$

7. Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$, given that $y = 3$, when $x = 0$

8. Solve the following differential equations :

(a) $\left(x^2 - yx^2\right)\dfrac{dy}{dx} + y^2 + xy^2 = 0$ (b) $\dfrac{dy}{dx} = xy + x + y + 1$

(c) $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$ (d) $\dfrac{dy}{dx} = e^{x-y} + e^{-y}x^2$

9. Solve the following differential equations :

(a) $\left(x^2 + y^2\right)dx - 2xy\,dy = 0$ (b) $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$

(c) $\dfrac{dy}{dx} = \dfrac{\sqrt{x^2 - y^2} + y}{x}$ (d) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$

10. Solve : $\dfrac{dy}{dx} + y \sec x = \tan x$

11. Solve the following differential equations :

(a) $\left(1 + x^2\right)\dfrac{dy}{dx} + y = \tan^{-1} x$ (b) $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

(c) $x \log x \dfrac{dy}{dx} + y = 2 \log x, \; x > 1$

12. Solve the following differential equations:

(a) $(x + y + 1)\dfrac{dy}{dx} = 1$

[Hint: $\dfrac{dx}{dy} = x + y + 1$ or $\dfrac{dx}{dy} - x = y + 1$ which is of the form $\dfrac{dx}{dy} + Px = Q$]

(b) $\left(x + 2y^2\right)\dfrac{dy}{dx} = y, \; y > 0$ [Hint: $y\dfrac{dx}{dy} = x + 2y^2$ or $\dfrac{dx}{dy} - \dfrac{x}{y} = 2y$]

Example 32.16 Verify if $y = e^{m \sin^{-1} x}$ is a solution of

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - \frac{dy}{dx} - m^2 y = 0$$

Solution : We have,

$$y = e^{m \sin^{-1} x} \qquad .....(1)$$

Differentiating (1) w.r.t. x , we get

$$\frac{dy}{dx} = \frac{me^{m \sin^{-1} x}}{\sqrt{1 - x^2}} = \frac{my}{\sqrt{1 - x^2}}$$

or $$\sqrt{1 - x^2}\,\frac{dy}{dx} = my$$

Squaring both sides, we get

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

Differentiating both sides, we get

$$-2x\left(\frac{dy}{dx}\right)^2 + 2\left(1-x^2\right)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 2m^2y\frac{dy}{dx}$$

or $$-x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = m^2y$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Hence $y = e^{m\sin^{-1}x}$ is the solution of

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Example 32.17 Find the equation of the curve represented by

$$(y - yx)\,dx + (x + xy)\,dy = 0$$

and passing through the point (1, 1).

Solution : The given differential equation is

$$(y - yx)\,dx + (x + xy)\,dy = 0$$

or $$(x + xy)\,dy = (yx - y)\,dx$$

or $$x(1+y)\,dy = y(x-1)\,dx$$

or $$\frac{(1+y)}{y}dy = \frac{x-1}{x}dx \qquad .....(1)$$

Integrating both sides of equation (1), we get

$$\int\left(\frac{1+y}{y}\right)dy = \int\left(\frac{x-1}{x}\right)dx$$

or $$\int\left(\frac{1}{y}+1\right)dy = \int\left(1-\frac{1}{x}\right)dx \qquad .....(2)$$

or $$\log y + y = x - \log x + C$$

Since the curve is passing through the point (1,1), therefore,

substituting x = 1, y = 1 in equation (2), we get

$$1 = 1 + C$$

or $$C = 0$$

Thus, the equation of the required curve is

$$\log y + y = x - \log x$$

or $$\log(xy) = x - y$$

Example 32.18 Solve $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$

Solution : We have $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$

or $$\frac{dy}{dx} = \frac{3e^{3x}\left(e^{-x} + e^x\right)}{e^x + e^{-x}}$$ or $$\frac{dy}{dx} = 3e^{3x}$$

or $$dy = 3e^{3x}dx \qquad .....(1)$$

Integrating both sides of (1), we get

$$y = \int 3e^{3x}dx + C$$

where C is an arbitrary constant.

or $$y = 3\frac{e^{3x}}{3} + C$$ or $$y = e^{3x} + C$$

which is required solution.

$$y(1 + ax)\left(1 - a^2\right) = x(1 - ay)\left(1 + a^2\right)$$

which is the required solution.

CHECK YOUR PROGRESS 32.3

1. (a) If $y = \tan^{-1} x$, prove that $\left(1 + x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

   (b) $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

2. (a) Find the equation of the curve represented by

   $\frac{dy}{dx} = xy + x + y + 1$ and passing through the point (2, 0)

   (b) Find the equation of the curve represented by

   $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$ and passing through the point $\left(\frac{\pi}{2}, 2\right)$

3. Solve : $\dfrac{dy}{dx} = \dfrac{4e^{3x} + 4e^{5x}}{e^{x} + e^{-x}}$

4. Solve the following differential equations :

(a) $dx + xdy = e^{-y} \sec^2 y \, dy$

(b) $\left(1 + x^2\right)\dfrac{dy}{dx} - 4x = 3\cot^{-1} x$

(c) $(1 + y)\, xy \, dy = \left(1 - x^2\right)(1 - y)\, dx$

LET US SUM UP

- A differential equation is an equation involving independent variable, dependent variable and the derivatives of dependent variable (and differentials) with respect to independent variable.
- The order of a differential equation is the order of the highest derivative occurring in it.
- The degree of a differential equation is the degree of the highest derivative.
- Degree of a differential equation exists, if it is a polynomial equation in terms of its derivatives.
- A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a linear differential equation.
- A linear differential equation is always of the first degree.
- A general solution of a differential equation is that solution which contains as many as the number of arbitrary constants as the order of the differential equation.
- A general solution becomes a particular solution when particular values of the arbitrary constants are determined satisfying the given conditions.
- The solution of the differential equation of the type $\dfrac{dy}{dx} = f(x)$ is obtained by integrating both sides.
- The solution of the differential equation of the type $\dfrac{dy}{dx} = f(x)\, g(y)$ is obtained after separating the variables and integrating both sides.
- The differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous and are of the same degree.
- The solution of a homogeneous differential equation is obtained by substituting $y = vx$ or $x = vy$ and then separating the variables.
- The solution of the first order linear equation $\dfrac{dy}{dx} + Py = Q$ is

$$ye^{\int P dx} = \int Q\left(e^{\int P dx}\right) dx + C,$$ where C is an arbitrary constant.

The expression $e^{\int P dx}$ is called the integrating factor of the differential equation and is written as I.F. in short.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=9Wfn-WWV1aY

http://www.youtube.com/watch?v=6YRGEsQWZzY

TERMINAL EXERCISE

1. Find the order and degree of the differential equation :

   (a) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^4 = 0$ (b) $xdx + ydy = 0$

   (c) $\frac{d^4y}{dx^4} - 4\frac{dy}{dx} + 4y = 5\cos 3x$ (d) $\frac{dy}{dx} = \cos x$

   (e) $x^2\frac{d^2y}{dx^2} - xy\frac{dy}{dx} = y$ (f) $\frac{d^2y}{dx^2} + y = 0$

2. Find which of the following equations are linear and which are non-linear

   (a) $\frac{dy}{dx} = \cos x$ (b) $\frac{dy}{dx} + \frac{y}{x} = y^2 \log x$

   (c) $\left(\frac{d^2y}{dx^2}\right)^3 + x^2\left(\frac{dy}{dx}\right)^2 = 0$ (d) $x\frac{dy}{dx} - 4 = x$

   (e) $dx + dy = 0$

3. Form the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.

4. Find the differential equation by eliminating a, b, c from

   $y = ax^2 + bx + c$. Write its order and degree.

5. How many constants are contained in the general solution of

   (a) Second order differential equation.

(b) Differential equation of order three.

(c) Differential equation of order five.

6. Show that $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

7. Solve the following differential equations:

(a) $\sin^2 x\dfrac{dy}{dx} = 3\cos x + 4$ (b) $\dfrac{dy}{dx} = e^{x-y} + x^2e^{-y}$

(c) $\dfrac{dy}{dx} + \dfrac{\cos x\sin y}{\cos y} = 0$ (d) $dy + xydx = xdx$

(e) $\dfrac{dy}{dx} + y\tan x = x^m\cos mx$ (f) $\left(1+y^2\right)\dfrac{dx}{dy} = \tan^{-1}y - x$

ANSWERS

CHECK YOUR PROGRESS 32.1

1. Order is 1 and degree is 1.
2. (a) Order 2, degree 1
   (b) Order 2, degree 2
3. (a) Non- linear (b) Linear
   (c) Linear (d) Non-linear
4. $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$
5. (a) $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$

   (b) $\left(x^2 - 2y^2\right)\left(\dfrac{dy}{dx}\right)^2 - 4xy\dfrac{dy}{dx} - x^2 = 0$

   (c) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$

   (d) $y = (x-3)\dfrac{dy}{dx} + 2$

   (e) $\left(x^2 - y^2\right)\dfrac{dy}{dx} - 2xy = 0$

CHECK YOUR PROGRESS 32.2

1. (i) Yes (ii) No
2. (i), (ii) and (iv) are particular solutions (iii) is the general solution
3. (ii), (iv) are homogeneous
5. (a) $y = \tan x$
   (b) $y = \tan x + \sqrt{3}$
6. (a) $y = \dfrac{1}{6}x^6\tan^{-1}\left(x^3\right) - \dfrac{1}{6}x^3 + \dfrac{1}{6}\tan^{-1}\left(x^3\right) + C$

   (b) $y = \dfrac{1}{5}\cos^5 x - \dfrac{1}{3}\cos^3 x + (x-1)e^x + C$

(c) $y = \frac{1}{2}\log\left|x^2+1\right| + C$ (d) $y = \frac{1}{3}x^3 - \frac{1}{3}\cos 3x + C$

7. $y = -4e^{-x} + 7$

8. (a) $\log\left|\frac{x}{y}\right| = C + \frac{1}{x} + \frac{1}{y}$ (b) $\log|y+1| = x + \frac{x^2}{2} + C$

(c) $\tan x \tan y = C$ (d) $e^y = e^x + \frac{x^3}{3} + C$

9. (a) $x = C\left(x^2 - y^2\right)$ (b) $x + cy = y \log|x|$

(c) $\sin^{-1}\left(\frac{y}{x}\right) = \log|x| + C$ (d) $\tan\frac{y}{2x} = Cx$

10. $y(\sec x + \tan x) = \sec x + \tan x - x + C$

11. (a) $y = \tan^{-1} x - 1 + Ce^{-\tan x}$ (b) $y = \tan x - 1 + Ce^{-\tan x}$

(c) $y = \log x + \frac{C}{\log x}$

12. (a) $x = Ce^y - (y+2)$ (b) $x = y^2 + Cy$

CHECK YOUR PROGRESS 32.3

2. (a) $\log(y+1) = \frac{1}{2}x^2 + x - 4$ (b) $y \sin x + 5e^{\cos x} = 7$

3. $y = \frac{4}{5}e^{5x} + C$

4. (a) $x = e^{-y}(C + \tan y)$ (b) $y = 2\log\left|1+x^2\right| - \frac{3}{2}\left(\cot^{-1} x\right)^2 + C$

(c) $\log x + 2\log|1-y| = \frac{x^2}{2} - \frac{y^2}{2} - 2y + C$

TERMINAL EXERCISE

1. (a) Order 2, degree 3 (b) Order 1, degree 1

(c) Order 4, degree 1 (d) Order 1, degree 1

(e) Order 2, degree 1 (f) Order 2, degree 1

2. (a), (d), (e) are linear; (b), (c) are non-linear

3. $\left(x^2 - 2y^2\right)\left(\frac{dy}{dx}\right)^2 - 4xy\left(\frac{dy}{dx}\right) - x^2 = 0$

4. $\frac{d^3y}{dx^3} = 0$, Order 3, degree 1.

5. (a) Two (b) Three

(c) Five

7. (a) $y + 3 \operatorname{cosec} x + 4 \cot x = C$ (b) $e^y = e^x + \frac{x^3}{3} + C$

(c) $\sin y = Ce^{-\sin x}$ (d) $\log(1 - y) + \frac{x^2}{2} = C$

(e) $y = \frac{x^{m+1}}{m+1} \cos x + C \cos x$ (f) $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$

33

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

You have read in your earlier lessons that given a point in a plane, it is possible to find two numbers, called its co-ordinates in the plane. Conversely, given any ordered pair (x, y) there corresponds a point in the plane whose co-ordinates are (x, y).

Let a rubber ball be dropped vertically in a room The point on the floor, where the ball strikes, can be uniquely determined with reference to axes, taken along the length and breadth of the room. However, when the ball bounces back vertically upward, the position of the ball in space at any moment cannot be determined with reference to two axes considered earlier. At any instant, the position of ball can be uniquely determined if in addition, we also know the height of the ball above the floor.

If the height of the ball above the floor is 2.5cm and the position of the point where it strikes the ground is given by (5, 4), one way of describing the position of ball in space is with the help of these three numbers (5, 4, 2.5).

Thus, the position of a point (or an article) in space can be uniquely determined with the help of three numbers. In this lesson, we will discuss in details about the co-ordinate system and co-ordinates of a point in space, distance between two points in space, position of a point dividing the join of two points in a given ratio internally/externally and about the projection of a point/line in space.

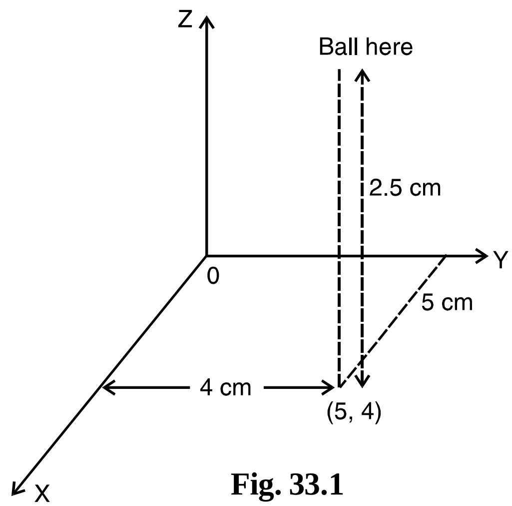


Fig. 33.1

OBJECTIVES

After studying this lesson, you will be able to :

- associate a point, in three dimensional space with given triplet and vice versa;
- find the distance between two points in space;
- find the coordinates of a point which divides the line segment joining twogiven points in a given ratio internally and externally;
- define the direction cosines/ratios of a given line in space;
- find the direction cosines of a line in space;

MODULE - IX
Vectors and three dimensional Geometry

Notes

- find the projection of a line segment on another line; and
- find the condition of prependicularity and parallelism of two lines in space.

EXPECTED BACKGROUND KNOWLEDGE

- Two dimensional co-ordinate geometry
- Fundamentals of Algebra, Geometry, Trigonometry and vector algebra.

33.1 COORDINATE SYSTEM AND COORDINATES OF A POINT IN SPACE

Recall the example of a bouncing ball in a room where one corner of the room was considered as the origin.

It is not necessary to take a particular corner of the room as the origin. We could have taken any corner of the room (for the matter any point of the room) as origin of reference, and relative to that the co-ordinates of the point change. Thus, the origin can be taken arbitarily at any point of the room.

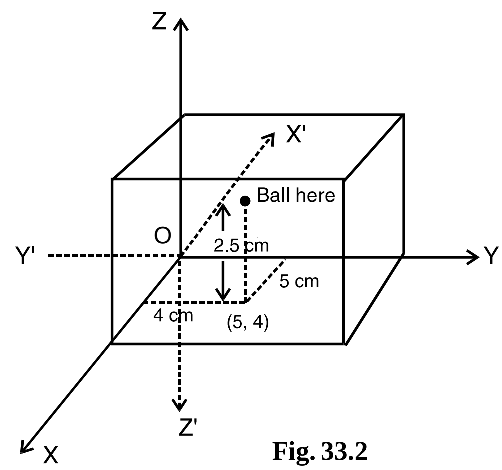


Fig. 33.2

Let us start with an arbitrary point O in space and draw three mutually perpendicular lines X'OX, Y'OY and Z'OZ through O. The point O is called the origin of the co-ordinate system and the lines X'OX, Y'OY and Z'OZ are called the X-axis, the Y-axis and the Z-axis respectively. The positive direction of the axes are indicated by arrows on thick lines in Fig. 33.2. The plane determined by the X-axis and the Y-axis is called XY-plane (XOY plane) and similarly, YZ-plane (YOZ-plane) and ZX-plane (ZOX-plane) can be determined. These three planes are called co-ordinate planes. The three coordinate planes divide the whole space into eight parts called **octants**.

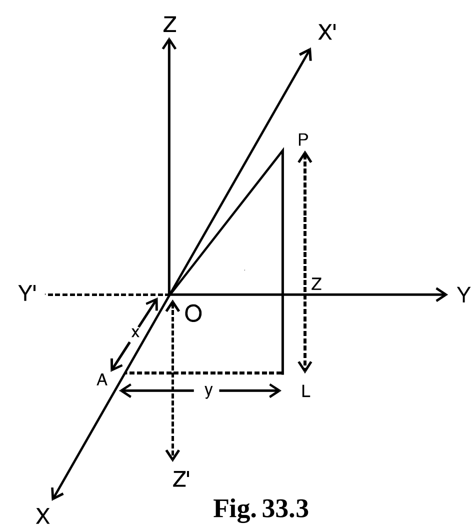


Fig. 33.3

Let P be any point is space. Through P draw perpendicular PL on XY-plane meeting this plane at L. Through L draw a line LA parallel to OY cutting OX in A. If we write OZ = x, AL = y and LP = z, then (x, y, z) are the co-ordinates of the point P.

Again, if we complete a reactangular parallelopiped through P with its three edges OA, OB and OC meeting each other at O and OP as its main diagonal then the lengths (OA, OB, OC) i.e., (x, y, z) are called the co-ordinates of the point P.

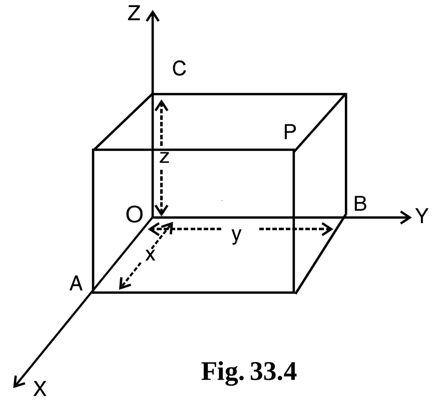


Fig. 33.4

Note : You may note that in Fig. 33.4

(i) The x co-ordinate of P = OA = the length of perpendicular from P on the YZ-plane.

(ii) The y co-ordinate of P = OB = the length of perpendicular from P on the ZX-plane.

(iii) The x co-ordinate of P = OC = the length of perpendicular from P on the XY-plane.

Thus, the co-ordinates x, y, and z of any point are the perpendicular distances of P from the three rectangular co-ordinate planes YZ, ZX and XY respectively.

Thus, given a point P in space, to it corresponds a triplet (x, y, z) called the co-ordinates of the point in space. Conversely, given any triplet (x, y, z), there corresponds a point P in space whose co-ordinates are (x, y, z).

Remarks

1. Just as in plane co-ordinate geometry, the co-ordinate axes divide the plane into four quadrants, in three dimentional geometry, the space is divided into eight octants by the co-ordinate planes, namely OXYZ, OX'YZ, OXY'Z, OXYZ', OXY'Z', OX'YZ', OX'Y'Z and OX'Y'Z'.

2. If P be any point in the first octant, there is a point in each of the other octants whose absolute distances from the co-ordinate planes are equal to those of P. If P be (a, b, c), the other points are (–a, b, c), (a, –b, c), (a, b, –c), (a, –b, –c), (–a, b, –c), (–a, –b, c) and (–a, –b, –c) respectively in order in the octants referred in (i).

3. The co-ordinates of point in XY-plane, YZ-plane and ZX-plane are of the form (a, b, 0), (0, b, c) and (a, 0, c) respectively.

4. The co-ordinates of points on X-axis, Y-axis and Z-axis are of the form (a, 0, 0), (0, b, 0) and (0, 0, c) respectively.

5. You may see that (x, y, z) corresponds to the position vector of the point P with reference to the origin O as the vector $\overrightarrow{OP}$.

Example 33.1 Name the octant wherein the given points lies :

(a) (2, 6, 8) (b) (–1, 2, 3) (c) (–2, –5, 1)

(d) $(-3, 1, -2)$ (e) $(-6, -1, -2)$

Solution :

(a) Since all the co-ordinates are positive, $\therefore$ (2, 6, 8) lies in the octant OXYZ.

(b) Since x is negative and y and z are positive, $\therefore$ (–1, 2, 3) lies in the octant OX'YZ.

(c) Since x and y both are negative and z is positive $\therefore$ (–2, –5, 1) lies in the octant OX'Y'Z.

(d) (–3, 1, –2) lies in octant OX'YZ'.

(e) Since x, y and z are all negative $\therefore$ (–6, –1, –2) lies in the octant OX'Y'Z'.

CHECK YOUR PROGRESS 33.1

1. Name the octant wherein the given points lies :

(a) $(-4, 2, 5)$ (b) $(4, 3, -6)$ (c) $(-2, 1, -3)$

(d) $(1, -1, 1)$ (f) $(8, 9, -10)$

33.2 DISTANCE BETWEEN TWO POINTS

Suppose there is an electric plug on a wall of a room and an electric iron placed on the top of a table. What is the shortest length of the wire needed to connect the electric iron to the electric plug ? This is an example necessitating the finding of the distance between two points in space.

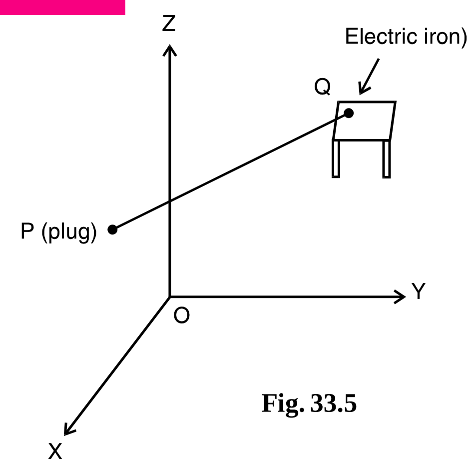


Fig. 33.5

Let the co-ordinates of two points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. With PQ as diagonal, complete the parallopiped PMSNRLKQ.

PK is perpendicular to the line KQ.

$\therefore$ From the right-angled triangle PKQ, right angled at K,

We have $PQ^2 = PK^2 + KQ^2$

Again from the right angled triangle PKL right angled at L,

$$PK^2 = KL^2 + PL^2 = MP^2 + PL^2 \quad (\because KL = MP)$$

$$\therefore \quad PQ^2 = MP^2 + PL^2 + KQ^2 \quad ...(i)$$

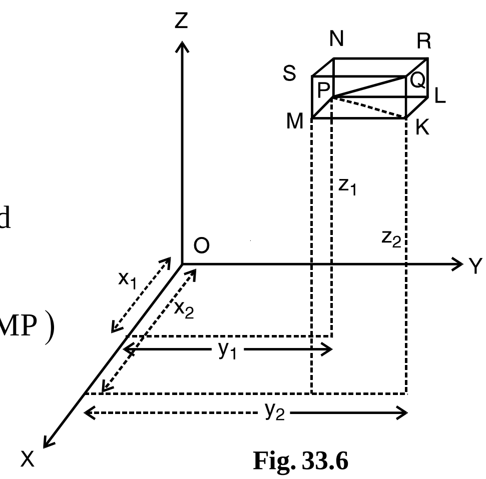


Fig. 33.6

The edges MP, PL and KQ are parallel to the co-ordinate axes.

Now, the distance of the point P from the plane $YOZ = x_1$ and the distance of Q and M from

YOZ plane $= x_2$

$\therefore \qquad MP = |x_2 - x_1|$

Similarly, $\qquad PL = |y_2 - y_1|$ and $KQ = |z_2 - z_1|$

$\therefore \qquad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ ....[From (i)]

or $\qquad \mathbf{|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$

Corollary : Distance of a Point from the Origin

If the point $Q(x_2, y_2, z_2)$ coincides with the origin (0, 0, 0), then $x_2 = y_2 = z_2 = 0$

$\therefore$ The distance of P from the origin is

$$|OP| = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2}$$

$$= \sqrt{x_1^2 + y_1^2 + z_1^2}$$

In general, the distance of a point P(x, y, z) from origin O is given by

$$|OP| = \sqrt{x^2 + y^2 + z^2}$$

Example 33.2 Find the distance between the points $(2, 5, -4)$ and $(8, 2, -6)$.

Solution : Let $P(2, 5, -4)$ and $Q(8, 2, -6)$ be the two given points.

$$\therefore \qquad |PQ| = \sqrt{(8-2)^2 + (2-5)^2 + (-6+4)^2}$$

$$= \sqrt{36 + 9 + 4}$$

$$= \sqrt{49}$$

$$= 7$$

$\therefore$ The distance between the given points is 7 units.

Example 33.3 Prove that the points $(-2, 4, -3)$, $(4, -3, -2)$ and $(-3, -2, 4)$ are the vertices of an equilateral triangle.

Solution : Let $A(-2, 4, -3)$, $B(4, -3, -2)$ and $C(-3, -2, 4)$ be the three given points.

Now $\qquad |AB| = \sqrt{(4+2)^2 + (-3-4)^2 + (-2+3)^2}$

$$= \sqrt{36 + 49 + 1} = \sqrt{86}$$

$$|BC| = \sqrt{(-3-4)^2 + (-2+3)^2 + (4+2)^2} = \sqrt{86}$$

$$|CA| = \sqrt{(-2+3)^2 + (4+2)^2 + (-3-4)^2} = \sqrt{86}$$

Since $|AB| = |BC| = |CA|$, $\Delta\ ABC$ is an equilateral triangle.

Example 33.4 Verify whether the following points form a triangle or not :

(a) $A(-1, 2, 3)$ $B(1, 4, 5)$ and $C(5, 4, 0)$

(b) $(2, -3, 3)$, $(1, 2, 4)$ and $(3, -8, 2)$

Solution :

(a) $$|AB| = \sqrt{(1+1)^2 + (4-2)^2 + (5-3)^2}$$

$$= \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3} \quad = 3.464 \text{ (approx.)}$$

$$|BC| = \sqrt{(5-1)^2 + (4-4)^2 + (0-5)^2}$$

$$= \sqrt{16 + 0 + 25} = \sqrt{41} \quad = 6.4 \text{ (approx.)}$$

and $$|AC| = \sqrt{(5+1)^2 + (4-2)^2 + (0-3)^2}$$

$$= \sqrt{36 + 4 + 9} = 7$$

$\therefore$ $|AB| + |BC| = 3.464 + 6.4 = 9.864 > |AC|$, $|AB| + |AC| > |BC|$

and $|BC| + |AC| > |AB|$.

Since sum of any two sides is greater than the third side, therefore the above points form a triangle.

(b) Let the points $(2, -3, 3), (1, 2, 4)$ and $(3, -8, 2)$ be denoted by P, Q and R respectively,

then $$|PQ| = \sqrt{(1-2)^2 + (2+3)^2 + (4-3)^2}$$

$$= \sqrt{1 + 25 + 1} = 3\sqrt{3}$$

$$|QR| = \sqrt{(3-1)^2 + (-8-2)^2 + (2-4)^2}$$

$$= \sqrt{4 + 100 + 4} = 6\sqrt{3}$$

$$|PR| = \sqrt{(3-2)^2 + (-8+3)^2 + (2-3)^2}$$

$$= \sqrt{1 + 25 + 1}$$

$$= 3\sqrt{3}$$

In this case $|PQ| + |PR| = 3\sqrt{3} + 3\sqrt{3} = 6\sqrt{3} = |QR|$. Hence the given points do not form a triangle. In fact the points lie on a line.

Example 33.5 Show that the points $A(1, 2, -2), B(2, 3, -4)$ and $C(3, 4, -3)$ form a right angled triangle.

Solution : $AB^2 = (2-1)^2 + (3-2)^2 + (-4+2)^2 = 1 + 1 + 4 = 6$

$BC^2 = (3-2)^2 + (4-3)^2 + (-3+4)^2 = 1 + 1 + 1 = 3$

and $AC^2 = (3-1)^2 + (4-2)^2 + (-3+2)^2 = 4 + 4 + 1 = 9$

We observe that $AB^2 + BC^2 = 6 + 3 = 9 = AC^2$

$\therefore$ ΔABC is a right angled triangle.

Hence the given points form a right angled triangle.

Example 33.6 Prove that the points $A(0, 4, 1)$, $B(2, 3, -1)$, $C(4, 5, 0)$ and $D(2, 6, 2)$ are vertices of a square.

Solution : Here,

$$AB = \sqrt{(2-0)^2 + (3-4)^2 + (-1-1)^2}$$

$$= \sqrt{4+1+4} = 3 \text{ units}$$

$$BC = \sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2}$$

$$= \sqrt{4+4+1} = 3 \text{ units}$$

$$CD = \sqrt{(2-4)^2 + (6-5)^2 + (2-0)^2}$$

$$= \sqrt{4+1+4} = 3 \text{ units}$$

and

$$DA = \sqrt{(0-2)^2 + (4-6)^2 + (1-2)^2}$$

$$= \sqrt{4+4+1} = 3 \text{ units}$$

$$\therefore \quad AB = BC = CD = DA$$

Now

$$AC^2 = \sqrt{(4-0)^2 + (5-4)^2 + (0-1)^2}$$

$$= 16 + 1 + 1 = 18$$

$$\therefore \quad AB^2 + BC^2 = 3^2 + 3^2 = 18 = AC^2$$

$$\therefore \quad \angle B = 90^\circ$$

$\therefore$ In quadrilateral A B C D, $AB = BC = CD = DA$ and $\angle B = 90^\circ$

$\therefore$ A B C D is a square.

CHECK YOUR PROGRESS 33.2

1. Find the distance between the following points :

   (a) $(4, 3, -6)$ and $(-2, 1, -3)$ (b) $(-3, 1, -2)$ and $(-3, -1, 2)$

   (c) $(0, 0, 0)$ and $(-1, 1, 1)$

2. Show that if the distance between the points $(5, -1, 7)$ and $(a, 5, 1)$ is 9 units, "a" must be either 2 or 8.

3. Show that the triangle formed by the points (a, b, c), (b, c, a) and (c, a, b) is equilateral.

4. Show that the the points $(-1, 0, -4)$, $(0, 1, -6)$ and $(1, 2, -5)$ form a right angled tringle.

5. Show that the points $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of an isosceles right-angled triangle.

6. Show that the points $(3, -1, 2)$, $(5, -2, -3)$, $(-2, 4, 1)$ and $(-4, 5, 6)$ form a parallelogram.

7. Show that the points $(2, 2, 2)$, $(-4, 8, 2)$, $(-2, 10, 10)$ and $(4, 4, 10)$ form a square.

8. Show that in each of the following cases the three points are collinear :

   (a) $(-3, 2, 4)$, $(-1, 5, 9)$ and $(1, 8, 14)$

   (b) $(5, 4, 2)$, $(6, 2, -1)$ and $(8, -2, -7)$

   (c) $(2, 5, -4)$, $(1, 4, -3)$ and $(4, 7, -6)$

33.3 COORDINATES OF A POINT OF DIVISION OF A LINE SEGMENT

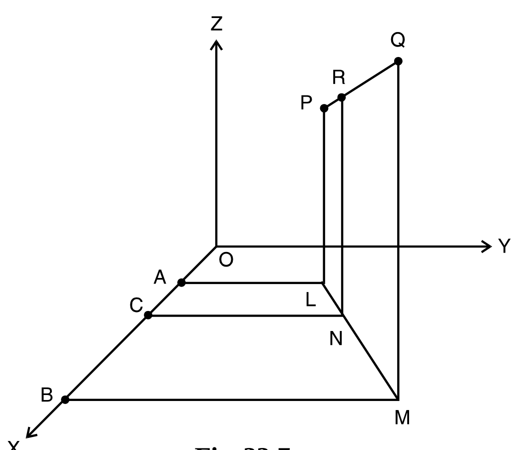


Fig. 33.7

Let the point R (x, y, z) divide PQ in the ratio $l : m$ internally.

Let the co-ordinates of P be (x_1, y_1, z_1) and the co-ordinates of Q be (x_2, y_2, z_2). From points P, R and Q, draw PL, RN and QM perpendiculars to the XY-plane.

Draw LA, NC and MB perpendiculars to OX.

Now $AC = OC - OA = x - x_1$ and $BC = OB - OC = x_2 - x$

Also we have, $\dfrac{AC}{BC} = \dfrac{LN}{NM} = \dfrac{PR}{RQ} = \dfrac{l}{m}$

$$\therefore \qquad \frac{x - x_1}{x_2 - x} = \frac{l}{m}$$

or $$mx - mx_1 = lx_2 - lx$$

or $$(l + m)x = lx_2 + mx_1$$

or $$x = \frac{lx_2 - mx_1}{l + m}$$

Similarly, if we draw perpendiculars to OY and OZ respectively,

we get $y = \dfrac{ly_2 + my_1}{l + m}$ and $z = \dfrac{lz_2 + mz_1}{l + m}$

$$\therefore \qquad \text{R is the point } \left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}, \frac{lz_2 + mz_1}{l + m} \right)$$

If $\lambda = \dfrac{l}{m}$, then the co-ordinates of the point R which divides PQ in the ratio $\lambda : 1$ are

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1} \right), \lambda + 1 \neq 0$$

It is clear that to every value of λ, there corresponds a point of the line PQ and to every point R on the line PQ, there corresponds some value of λ. If λ is postive, R lies on the line segment PQ and if λ is negative, R does not lie on line segment PQ.

In the second case you may say the R divides the line segment PQ externally in the ratio $-\lambda : 1$.

Corollary 1 : The co-ordinates of the point dividing PQ externally in the ratio $l : m$ are

$$\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m} \right)$$

Corollary 2 : The co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Example 33.7 Find the co-ordinates of the point which divides the line segment joining the points $(2, -4, 3)$ and $(-4, 5, -6)$ in the ratio 2 : 1 internally.

Solution : Let $A(2, -4, 3)$, $B(-4, 5, -6)$ be the two points.

Let P (x, y, z) divides AB in the ratio 2 : 1.

$$\therefore \qquad x = \frac{2(-4)+1.2}{2+1} = -2, \quad y = \frac{2.5+1(-4)}{2+1} = 2$$

$$\text{and} \qquad z = \frac{2(-6)+1.3}{2+1} = -3$$

Thus, the co-ordinates of P are $(-2, 2, -3)$

Example 33.8 Find the point which divides the line segment joining the points $(-1, -3, 2)$ and $(1, -1, 2)$ externally in the ratio 2 : 3.

Solution : Let the points $(-1, -3, 2)$ and $(1, -1, 2)$ be denoted by P and Q respectively.

Let R (x, y, z) divide PQ externally in the ratio 2 : 3. Then

$$x = \frac{2(1)-3(-1)}{2-3} = -5, \qquad y = \frac{2(-1)-3(-3)}{2-3} = -7$$

$$\text{and} \qquad z = \frac{2(2)-3(2)}{2-3} = 2$$

Thus, the co-ordinates of R are $(-5, -7, 2)$.

Example 33.9 Find the ratio in which the line segment joining the points $(2\ -3, 5)$ and $(7\ , 1, 3)$ is divided by the XY-plane.

Solution : Let the required ratio in which the line segment is divided be $l : m$.

The co-ordinates of the point are $\left(\frac{7l+2m}{l+m}, \frac{l-3m}{l+m}, \frac{3l+5m}{l+m}\right)$

Since the point lies in the XY-plane, its z-coordinate is zero.

$$\text{i.e.,} \qquad \frac{3l+5m}{l+m} = 0 \text{ or } \frac{l}{m} = -\frac{5}{3}$$

Hence the XY-plane divides the join of given points in the ratio 5 : 3 externally.

CHECK YOUR PROGRESS 33.3

1. Find the co-ordinates of the point which divides the line segment joining two points $(2, -5, 3)$ and $(-3, 5, -2)$ internally in the ratio 1 : 4.
2. Find the coordinates of points which divide the join of the points $(2, -3, 1)$ and $(3, 4, -5)$ internally and externally in the ratio 3 : 2.
3. Find the ratio in which the line segment joining the points $(2, 4, 5)$ and $(3, 5, -4)$ is divided by the YZ-plane.
4. Show that the YZ-plane divides the line segment joining the points $(3, 5, -7)$ and $(-2, 1, 8)$ in the ration 3 : 2 at the point $\left(0, \frac{13}{5}, 2\right)$.
5. Show that the ratios in which the co-ordinate planes divide the join of the points $(-2, 4, 7)$ and $(3, -5, 8)$ are 2 : 3, 4 : 5 (internally) and 7 : 8 (externally).
6. Find the co-ordinates of a point R which divides the line segment $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio 2 : 1. Verify that Q is the mid-point of PR.

LET US SUM UP

- For a given point P (x, y, z) in space with reference to reactangular co-ordinate axes, if we draw three planes parallel to the three co-ordinate planes to meet the axes (in A, B and C say), then
  OA = x, OB = y and OC = z where O is the origin.
  Converswly, given any three numbers, x, y and z we can find a unique point in space whose co-ordinates are (x, y, z).
- The distance PQ between the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

  In particular the distance of P from the origin O is $\sqrt{x_1^2 + y_1^2 + z_1^2}$.
- The co-ordinates of the point which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $l : m$

  (a) internally are $\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}, \frac{lz_2 + mz_1}{l + m}\right)$

  (b) externally are $\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m}\right)$

  In particular, the co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

SUPPORTIVE WEB SITES

http://www.mathguru.com/level3/introduction-to-three-dimensional-geometry-

http://www.goiit.com/posts/show/0/content-3-d-geometry-804299.htm

http://www.askiitians.com/iit-jee-3d-geometry

TERMINAL EXERCISE

1. Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form an isosceles right-angled triangle.

2. Prove that the points P, Q and R, whose co-ordinates are respectively $(3, 2, -4)$, (5, 4, –6) and (9, 8, –10) are collinear and find the ratio in which Q divides PR.

3. Show that the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

4. Show that the points (4, 7, 8), (2, 3, 4), (–1, –2, 1) and (1, 2, 5) are the vertices of a parallelogram.

5. Three vertices of a parallelogram ABCD are A (3, –4, 7), B (5, 3, –2) and C (1, 2, –3). Find the fourth vertex D.

ANSWERS

CHECK YOUR PROGRESS 33.1

1. (a) OX' YZ (b) OXYZ' (c) OX' YZ'
   (d) OXY'Z (e) OXYZ'

CHECK YOUR PROGRESS 33.2

1. (a) 7 (b) $2\sqrt{5}$ (c) $\sqrt{3}$

CHECK YOUR PROGRESS 33.3

1. $(1, -3, 2)$ 2. $\left(\frac{13}{5}, \frac{6}{5}, -\frac{13}{5}\right)$; $(5, 18, -17)$

3. –2 : 3 6. $(2x_2 - x_1, 2y_2 - y_1, 2z_2 - z_1)$

TERMINAL EXERCISE

2. 1 : 2 5. $(-1, -5, 6)$

34

VECTORS

In day to day life situations, we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively. We also come across physical quantities such as dispacement, velocity, acceleration, momentum etc. which are of a difficult type.

Let us consider the following situation. Let A, B, C and D be four points equidistant (say 5 km each) from a fixed point P. If you are asked to travel 5 km from the fixed point P, you may reach either A, B, C, or D. Therefore, only starting (fixed point) and distance covered are not sufficient to describe the destination. We need to specify end point (terminal point) also. This idea of terminal point from the fixed point gives rise to the need for direction.

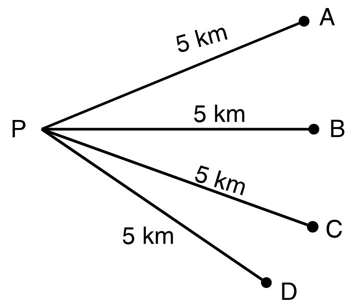


Fig. 34.1

Consider another example of a moving ball. If we wish to predict the position of the ball at any time what are the basics we must know to make such a prediction?

Let the ball be initially at a certain point A. If it were known that the ball travels in a straight line at a speed of 5cm/sec, can we predict its position after 3 seconds ? Obviously not. Perhaps we may conclude that the ball would be 15 cm away from the point A and therefore it will be at some point on the circle with A as its centre and radius 15 cms. So, the mere knowledge of speed and time taken are not sufficient to predict the position of the ball. However, if we know that the ball moves in a direction due east from A at a speed of 5cm/sec., then we shall be able to say that after 3 seconds, the ball must be precisely at the point P which is 15 cms in the direction east of A.

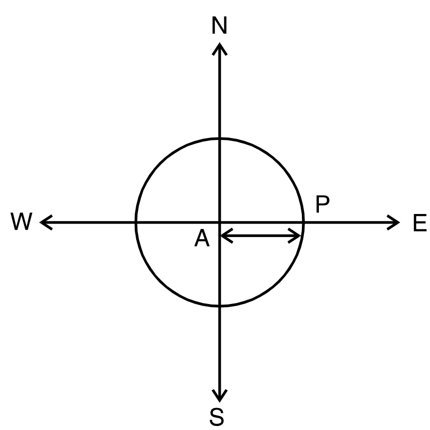


Fig. 34.2

Thus, to study the displacement of a ball after time t (3 seconds), we need to know the magnitude of its speed (i.e. 5 cm/sec) and also its direction (east of A)

In this lesson we will be dealing with quantities which have magnitude only, called scalars and the quantities which have both magnitude and direction, called vectors. We will represent vectors as directed line segments and determine their magnitudes and directions. We will study about various types of vectors and perform operations on vectors with properties thereof. We will also acquaint ourselves with position vector of a point w.r.t. some origin of reference. We will find out the resolved parts of a vector, in two and three dimensions, along two and three mutually perpendicular directions respectively. We will also derive section formula and apply that to problems. We will also define scalar and vector products of two vectors.

OBJECTIVES

After studying this lesson, you will be able to :

- explain the need of mentioning direction;
- define a scalar and a vector;
- distinguish between scalar and vactor;
- represent vectors as directed line segment;
- determine the magnitude and direction of a vector;
- classify different types of vectors-null and unit vectors;
- define equality of two vectors;
- define the position vector of a point;
- add and subtract vectors;
- multiply a given vector by a scalar;
- state and use the properties of various operations on vectors;
- comprehend the three dimensional space;
- resolve a vector along two or three mutually prependicular axes;
- derive and use section formula; and
- define scalar (dot) and vector (cross) product of two vectors.
- define and understand direction cosines and direction ratios of a vector.
- define triple product of vectors.
- understand scalar triple product of vectors and apply it to find volume of a rectangular parallelopiped.
- understand coplanarity of four points.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of plane and coordinate geometry.
- Knowledge of Trigonometry.

34.1 SCALARS AND VECTORS

A physical quantity which can be represented by a number only is known as a scalar i.e, quantities which have only magnitude. Time, mass, length, speed, temperature, volume, quantity of heat, work done etc. are all ***scalars***.

The physical quantities which have magnitude as well as direction are known as vectors. Displacement, velocity, acceleration, force, weight etc. are all examples of ***vectors***.

34.2 VECTOR AS A DIRECTED LINE SEGMENT

You may recall that a line segment is a portion of a given line with two end points. Take any line *l* (called a support). The portion of L with end points A and B is called a line segment. The line segment AB along with direction from A to B is written as $\overrightarrow{AB}$ and is called a directed line segment. A and B are respectively called the initial point and terminal point of the vector $\overrightarrow{AB}$.

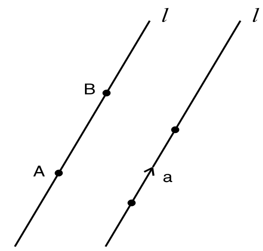


Fig. 34.3

The length AB is called the ***magnitude*** or ***modulus*** of $\overrightarrow{AB}$ and is denoted by $|\overrightarrow{AB}|$. In other words the length $AB = |\overrightarrow{AB}|$.

Scalars are usually represented by a, b, c etc. whereas vectors are usually denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$ etc. Magnitude of a vector $\vec{a}$ i.e., $|\vec{a}|$ is usually denoted by 'a'.

34.3 CLASSIFICATION OF VECTORS

34.3.1 Zero Vector (Null Vector)

A vector whose magnitude is zero is called a zero vector or ***null vector***. Zero vector has not definite direction. $\overrightarrow{AA}$, $\overrightarrow{BB}$ are zero vectors. Zero vectors is also denoted by $\vec{0}$ to distinguish it from the scalar 0.

34.3.2 Unit Vector

A vector whose magnitude is unity is called a ***unit vector***. So for a unit vector $\vec{a}$, $|\vec{a}| = 1$. A unit vector is usually denoted by $\hat{a}$. Thus, $\vec{a} = |\vec{a}|\,\hat{a}$.

34.3.3 Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude. i.e., $|\vec{a}| = |\vec{b}|$ and the same direction as shown in Fig. 14.4. Symbolically, it is denoted by $\vec{a} = \vec{b}$.

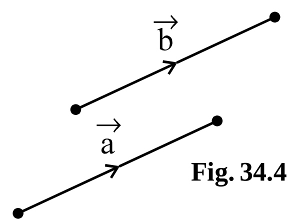


Fig. 34.4

Remark : Two vectors may be equal even if they have different parallel lines of support.

34.3.4 Like Vectors

Vectors are said to be like if they have same direction whatever be their magnitudes. In the adjoining Fig. 14.5, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are like vectors, although their magnitudes are not same.

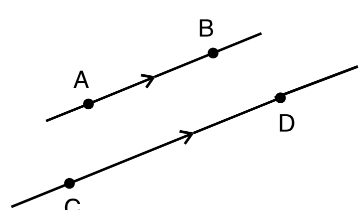


Fig. 34.5

MODULE - IX
Vectors and three dimensional Geometry

Notes

34.3.5 Negative of a Vector

$\overrightarrow{BA}$ is called the ***negative of the vector*** $\overrightarrow{AB}$, when they have the same magnitude but opposite directions.

i.e. $\overrightarrow{BA} = -\overrightarrow{AB}$

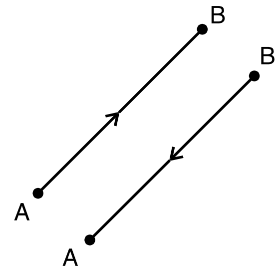


Fig. 34.6

34.3.6 Co-initial Vectors

Two or more vectors having the same initial point are called ***Co-initial vectors***.

In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AC}$ are co-initial vectors with the same initial point A.

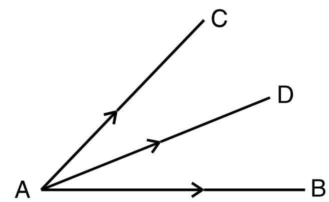


Fig. 34.7

34.3.7 Collinear Vectors

Vectors are said to be collinear when they are parallel to the same line whatever be their magnitudes. In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are collinear vectors. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are also ***collinear***.

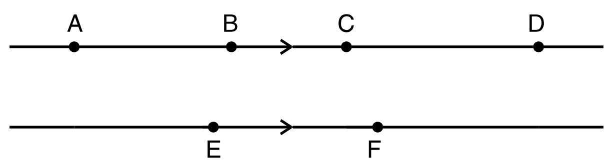


Fig. 34.8

34.3.8 Co-planar Vectors

Vectors are said to be co-planar when they are parallel to the same plane. In the adjoining figure $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-planar. Whereas $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie on the same plane, $\vec{d}$ is parallel to the plane of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

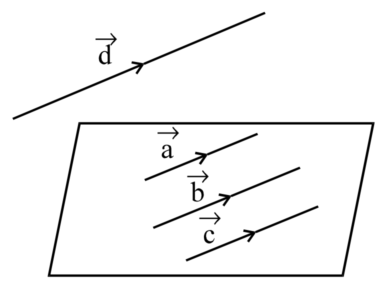


Fig. 34.9

Note : (i) A zero vector can be made to be collinear with any vector.

(ii) Any two vectors are always co-planar.

Example 34.1 State which of the following are scalars and which are vectors. Give reasons.

(a)	Mass	(b)	Weight	(c)	Momentum
(d)	Temperature	(e)	Force	(f)	Density

Solution : (a), (d) and (f) are scalars because these have only magnitude while (b), (c) and (e) are vectors as these have magnitude and direction as well.

Example 34.2 Represent graphically

(a) a force 40N in a direction 60° north of east.

(b) a force of 30N in a direction 40° east of north.

Solution :

(a)

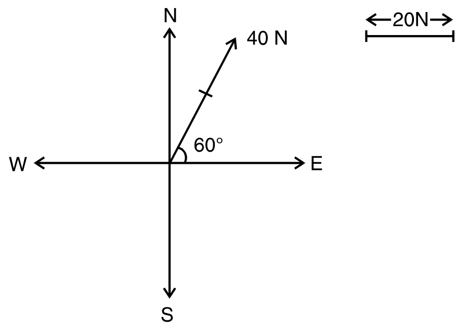


Fig. 34.10

(b)

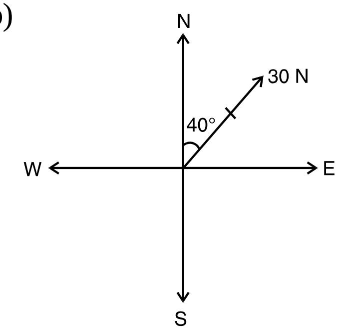


Fig. 34.11

CHECK YOUR PROGRESS 34.1

1. Which of the following is a scalar quantity ?

   (a) Displacement (b) Velocity (c) Force (d) Length.

2. Which of the following is a vector quantity ?

   (a) Mass (b) force (c) time (d) tempertaure

3. You are given a displacement vector of 5 cm due east. Show by a diagram the corresponding negative vector.

4. Distinguish between like and equal vectors.

5. Represent graphically

   (a) a force 60 Newton is a direction 60° west of north.

   (b) a force 100 Newton in a direction 45° north of west.

34.4 ADDITION OF VECTORS

Recall that you have learnt four fundamental operations viz. addition, subtraction, multiplication and division on numbers. The addition (subtraction) of vectors is different from that of numbers (scalars).

In fact, there is the concept of resultant of two vectors (these could be two velocities, two forces etc.) We illustrate this with the help of the following example :

Let us take the case of a boat-man trying to cross a river in a boat and reach a place directly in the line of start. Even if he starts in a direction perpendicular to the bank, the water current carries him to a place different from the place he desired., which is an example of the effect of two velocities resulting in a third one called the resultant velocity.

Thus, two vectors with magnitudes 3 and 4 may not result, on addition, in a vector with magnitude 7. It will depend on the direction of the two vectors i.e., on the angle between them. The addition of vectors is done in accordance with the triangle law of addition of vectors.

34.4.1 Triangle Law of Addition of Vectors

A vector whose effect is equal to the resultant (or combined) effect of two vectors is defined as the resultant or sum of these vectors. This is done by the triangle law of addition of vectors.

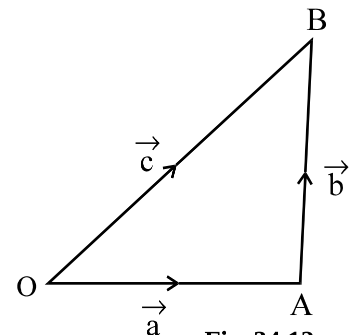


Fig. 34.12

In the adjoining Fig. 32.12 vector $\overrightarrow{OB}$ is the resultant or sum of vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ and is written as

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

i.e. $$\vec{a} + \vec{b} = \overrightarrow{OB} = \vec{c}$$

You may note that the terminal point of vector $\vec{a}$ is the initial point of vector $\vec{b}$ and the initial point of $\vec{a} + \vec{b}$ is the initial point of $\vec{a}$ and its terminal point is the terminal point of $\vec{b}$.

34.4.2 Addition of more than two Vectors

Addition of more then two vectors is shown in the adjoining figure

$$\vec{a} + \vec{b} + \vec{c} + \vec{d}$$
$$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$$
$$= \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE}$$
$$= \overrightarrow{AD} + \overrightarrow{DE}$$
$$= \overrightarrow{AE}$$

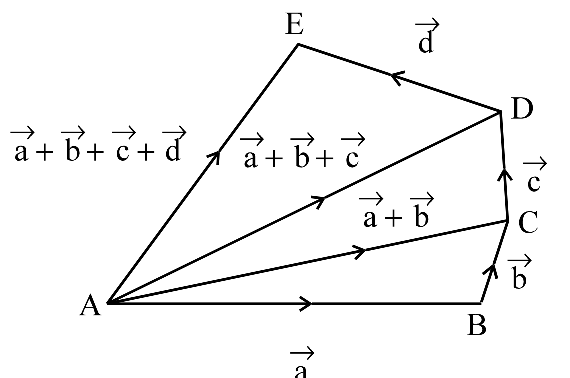


Fig. 34.13

The vector $\overrightarrow{AE}$ is called the sum or the resultant vector of the given vectors.

34.4.3 Parallelogram Law of Addition of Vectors

Recall that two vectors are equal when their magnitude and direction are the same. But they could be parallel [refer to Fig. 14.14].

See the parallelogram OABC in the adjoining figure :

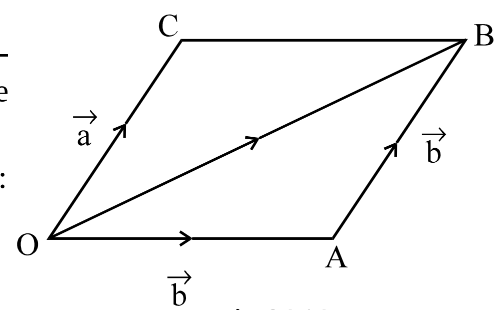


Fig. 34.14

We have, $$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

But $$\overrightarrow{AB} = \overrightarrow{OC}$$

$$\therefore \quad \overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$$

which is the parallelogram law of addition of vectors. **If two vectors are represented by the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal through the common point of the adjacent sides.**

34.4.4 Negative of a Vector

For any vector $\vec{a} = \overrightarrow{OA}$, the negative of $\vec{a}$ is represented by $\overrightarrow{AO}$. The negative of $\overrightarrow{AO}$ is the same as $\overrightarrow{OA}$. Thus, $|\overrightarrow{OA}| = |\overrightarrow{AO}| = |\vec{a}|$ and $\overrightarrow{OA} = -\overrightarrow{AO}$. It follows from definition that for any vector $\vec{a}$, $\vec{a} + (-\vec{a}) = \vec{0}$.

34.4.5 The Difference of Two Given Vectors

For two given vectors $\vec{a}$ and $\vec{b}$, the difference $\vec{a} - \vec{b}$ is defined as the sum of $\vec{a}$ and the negative of the vector $\vec{b}$. i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

In the adjoining figure if $\overrightarrow{OA} = \vec{a}$ then, in the parallelogram OABC, $\overrightarrow{CB} = \vec{a}$

and $\quad \overrightarrow{BA} = -\vec{b}$

$\therefore \quad \overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA} = \vec{a} - \vec{b}$

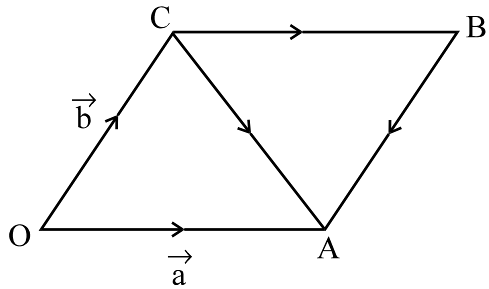


Fig. 34.15

Example 34.3 When is the sum of two non-zero vectors zero ?

Solution : The sum of two non-zero vectors is zero when they have the same magnitude but opposite direction.

Example 34.4 Show by a diagram $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Solution : From the adjoining figure, resultant

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$= \vec{a} + \vec{b} \qquad \text{....(i)}$$

Complete the parallelogram OABC

$$\overrightarrow{OC} = \overrightarrow{AB} = \vec{b},\ \overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$$

$$\therefore \quad \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$$

$$= \vec{b} + \vec{a} \qquad \text{....(ii)}$$

$$\therefore \quad \vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad [\text{ From (i) and (ii) }]$$

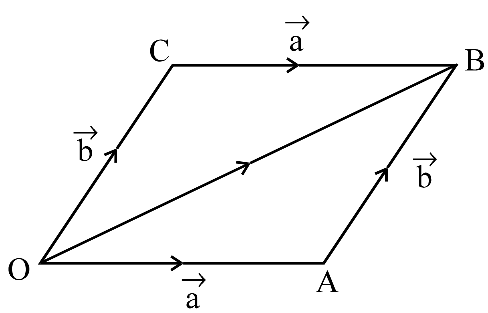


Fig. 34.16

CHECK YOUR PROGRESS 34.2

1. The diagonals of the parallelogram ABCD intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$.

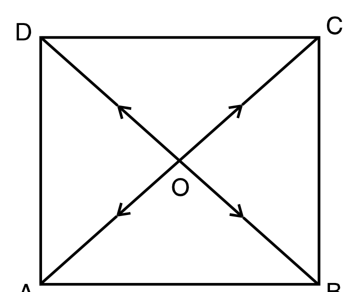


Fig. 34.17

MODULE - IX
Vectors and three dimensional Geometry

Notes

2. The medians of the triangle ABC intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

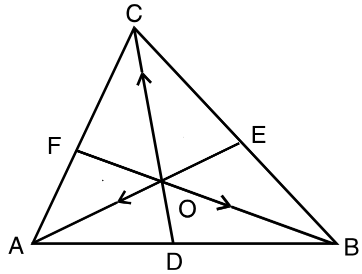


Fig. 34.18

34.5 POSITION VECTOR OF A POINT

We fix an arbitrary point O in space. Given any point P in space, we join it to O to get the vector $\overrightarrow{OP}$. This is called the position vector of the point P with respect to O, called the ***origin of reference***. Thus, to each given point in space there corresponds a unique position vector with respect to a given origin of reference. Conversely, given an origin of reference O, to each vector with the initial point O, corresponds a point namely, its terminal point in space.

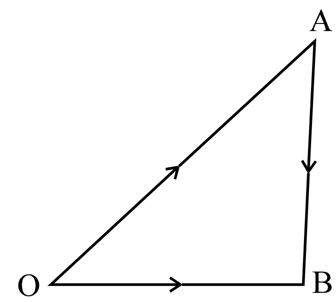


Fig. 34.19

Consider a vector AB. Let O be the origin of reference.

Then $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

or $\overrightarrow{AB}$ = (Position vector of terminal point B)–(Position vector of initial point A)

34.6 MULTIPLICATION OF A VECTOR BY A SCALAR

The product of a non-zero vector $\vec{a}$ by the scalar $x \neq 0$ is a vector whose length is equal to $|x||\vec{a}|$ and whose direction is the same as that of $\vec{a}$ if $x > 0$ and opposite to that of $\vec{a}$ if $x < 0$.

The product of the vector $\vec{a}$ by the scalar x is denoted by $x\vec{a}$.

The product of vector $\vec{a}$ by the scalar 0 is the vector $\vec{\mathbf{0}}$.

By the definition it follows that the product of a zero vector by any non-zero scalar is the zero vector i.e., $x\vec{0} = \vec{0}$; also $0\vec{a} = \vec{0}$.

Laws of multiplication of vectors : If $\vec{a}$ and $\vec{b}$ are vectors and x, y are scalars, then

(i) $x(y\vec{a}) = (xy)\vec{a}$

(ii) $x\vec{a} + y\vec{a} = (x+y)\vec{a}$

(iii) $x\vec{a} + x\vec{b} = x(\vec{a} + \vec{b})$

(iv) $0\vec{a} + x\vec{0} = \vec{0}$

Recall that two collinear vectors have the same direction but may have different magnitudes.

This implies that $\vec{a}$ is collinear with a non-zero vector $\vec{b}$ if and only if there exists a number (scalar) x such that

$$\vec{a} = x\vec{b}$$

Theorem A necessary and sufficient condition for two vectors $\vec{a}$ and $\vec{b}$ to be collinear is that there exist scalars x and y (not both zero simultaneously) such that $x\vec{a} + y\vec{b} = \vec{0}$.

The Condition is necessary

Proof : Let $\vec{a}$ and $\vec{b}$ be collinear. Then there exists a scalar l such that $\vec{a} = l\,\vec{b}$

i.e., $$\vec{a} + (-l)\,\vec{b} = \vec{0}$$

$\therefore$ We are able to find scalars x (= 1) and $y\ (= -l)$ such that $x\vec{a} + y\vec{b} = \vec{0}$

Note that the scalar 1 is non-zero.

The Condition is sufficient

It is now given that $x\vec{a} + y\vec{b} = \vec{0}$ and $x \neq 0$ and $y \neq 0$ simultaneously.

We may assume that $y \neq 0$

$\therefore$ $y\vec{b} = -x\vec{a} \Rightarrow \vec{b} = -\dfrac{x}{y}\vec{a}$ i.e., $\vec{b}$ and $\vec{a}$ are collinear.

Corollary : Two vectors $\vec{a}$ and $\vec{b}$ are non-collinear if and only if every relation of the form $x\vec{a} + y\vec{b} = \vec{0}$ given as x = 0 and y = 0.

[**Hint :** If any one of x and y is non-zero say y, then we get $\vec{b} = -\dfrac{x}{y}\vec{a}$ which is a contradiction]

Example 34.5 Find the number x by which the non-zero vector $\vec{a}$ be multiplied to get

(i) $\hat{a}$ (ii) $-\hat{a}$

Solution : (i) $x\vec{a} = \hat{a}$ i.e., $x\,|\vec{a}|\,\hat{a} = \hat{a}$

$$\Rightarrow \quad x = \frac{1}{|\vec{a}|}$$

(ii) $x\vec{a} = -\hat{a}$ i.e., $x\,|\vec{a}|\,\hat{a} = -\hat{a}$

$$\Rightarrow \quad x = -\frac{1}{|\vec{a}|}$$

Example 34.6 The vectors $\vec{a}$ and $\vec{b}$ are not collinear. Find x such that the vector

$$\vec{c} = (x-2)\vec{a} + \vec{b} \text{ and } \vec{d} = (2x+1)\vec{a} - \vec{b}$$

Solution : $\vec{c}$ is non-zero since the co-efficient of $\vec{b}$ is non-zero.

$\therefore$ There exists a number y such that $\vec{d} = y\vec{c}$

i.e. $$(2x+1)\vec{a} - \vec{b} = y(x-2)\vec{a} + y\vec{b}$$

$\therefore$ $$(yx - 2y - 2x - 1)\vec{a} + (y+1)\vec{b} = 0$$

As $\vec{a}$ and $\vec{b}$ are non-collinear.

$\therefore \quad yx - 2y - 2x - 1 = 0$ and $y + 1 = 0$

Solving these we get $y = -1$ and $x = \frac{1}{3}$

Thus $\quad \vec{c} = -\frac{5}{3}\vec{a} + \vec{b}$ and $\vec{d} = \frac{5}{3}\vec{a} - \vec{b}$

We can see that $\vec{c}$ and $\vec{d}$ are opposite vectors and hence are collinear.

Example 34.7 The position vectors of two points A and B are $2\vec{a} + 3\vec{b}$ and $3\vec{a} + \vec{b}$ respectively. Find $\overrightarrow{AB}$.

Solution : Let O be the origin of reference.

Then $\quad \overrightarrow{AB}$ = Position vector of B — Position vector of A

$$= \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (3\vec{a} + \vec{b}) - (2\vec{a} + 3\vec{b})$$

$$= (3 - 2)\vec{a} + (1 - 3)\vec{b} = \vec{a} - 2\vec{b}$$

Example 34.8 Show that the points P, Q and R with position vectors $\vec{a} - 2\vec{b}$, $2\vec{a} + 3\vec{b}$ and $-7\vec{b}$ respectively are collinear.

Solution : $\overrightarrow{PQ}$ = Position vector of Q — Position vector of P

$$= (2\vec{a} + 3\vec{b}) - (\vec{a} - 2\vec{b})$$

$$= \vec{a} + 5\vec{b} \qquad \text{....(i)}$$

and $\overrightarrow{QR}$ = Position vector of R — Position vector of Q

$$= -7\vec{b} - (2\vec{a} + 3\vec{b})$$

$$= -7\vec{b} - 2\vec{a} - 3\vec{b}$$

$$= -2\vec{a} - 10\vec{b}$$

$$= -2(\vec{a} + 5\vec{b}) \qquad \text{....(ii)}$$

From (i) and (ii) we get $\overrightarrow{PQ} = -2\,\overrightarrow{QR}$, a scalar multiple of $\overrightarrow{QR}$

$\therefore \quad \overrightarrow{PQ} \,||\, \overrightarrow{QR}$

But Q is a common point

$\therefore \quad \overrightarrow{PQ}$ and $\overrightarrow{QR}$ are collinear. Hence points P, Q and R are collinear.

CHECK YOUR PROGRESS 34.3

1. The position vectors of the points A and B are $\vec{a}$ and $\vec{b}$ respectively with respect to a given origin of reference. Find $\overrightarrow{AB}$.

2. Interpret each of the following :

   (i) $3\vec{a}$ (ii) $-5\vec{b}$

3. The position vectors of points A, B, C and D are respectively $2\vec{a}$, $3\vec{b}$, $4\vec{a}+3\vec{b}$ and $\vec{a}+2\vec{b}$. Find $\overrightarrow{DB}$ and $\overrightarrow{AC}$.

4. Find the magnitude of the product of a vector $\vec{n}$ by a scalar y.

5. State whether the product of a vector by a scalar is a scalar or a vector.

6. State the condition of collinearity of two vectors $\vec{p}$ and $\vec{q}$.

7. Show that the points with position vectors $5\vec{a}+6\vec{b}$, $7\vec{a}-8\vec{b}$ and $3\vec{a}+20\vec{b}$ are collinear.

34.7 CO-PLANARITY OF VECTORS

Given any two non-collinear vectors $\vec{a}$ and $\vec{b}$, they can be made to lie in one plane. There (in the plane), the vectors will be intersecting. We take their common point as O and let the two vectors be $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Given a third vector $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, we can choose its initial point also as O. Let C be its terminal point. With $\overrightarrow{OC}$ as diagonal complete the parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides.

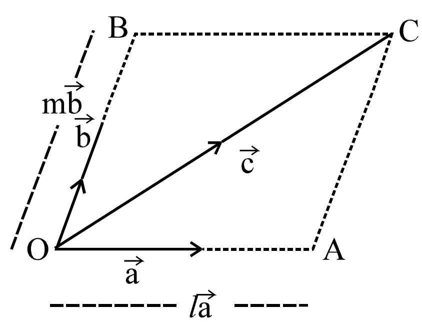


Fig. 34.20

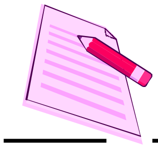
$$\therefore \qquad \vec{c} = l\vec{a} + m\vec{b}$$

Thus, any $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, is expressible as a linear combination of $\vec{a}$ and $\vec{b}$.

i.e. $$\vec{c} = l\vec{a} + m\vec{b}.$$

34.8 RESOLUTION OF A VECTOR ALONG TWO PER PERPEN DICULAR AXES

Consider two mutually perpendicular unit vectors $\hat{i}$ and $\hat{j}$ along two mutually perpendicular axes OX and OY. We have seen above that any vector $\vec{r}$ in the plane of $\hat{i}$ and $\hat{j}$, can be written in the form $\vec{r} = x\hat{i} + y\hat{j}$



If O is the initial point of $\vec{r}$, then OM = x and ON = y and $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are called the component vectors of $\vec{r}$ along x-axis and y-axis.

$\overrightarrow{OM}$ and $\overrightarrow{ON}$, in this special case, are also called the ***resolved parts*** of $\vec{r}$

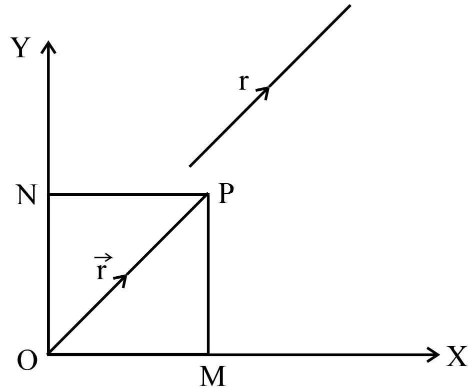


Fig. 34.21

34.9 RESOLUTION OF A VECTOR IN THREE DIMENSIONS ALONG THREE MUTUALLY PERPENDICULAR AXES

The concept of resolution of a vector in three dimensions along three mutually perpendicular axes is an extension of the resolution of a vector in a plane along two mutually perpendicular axes.

Any vector $\vec{r}$ in space can be expressed as a linear combination of three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ as is shown in the adjoining Fig. 14.22. We complete the rectangular parallelopiped with $\overrightarrow{OP} = \vec{r}$ as its diagonal :

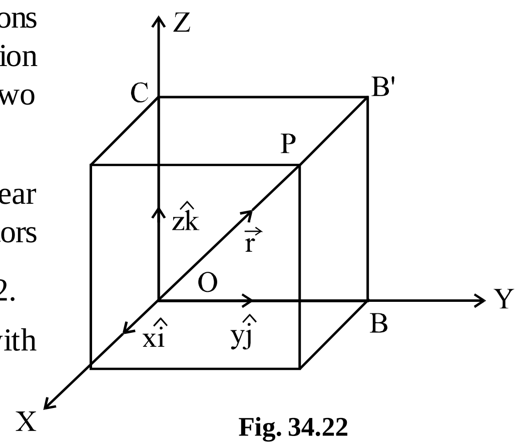


Fig. 34.22

then $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the resolved parts of $\vec{r}$ along three mutually perpendicular axes.

Thus any vector $\vec{r}$ in space is expressible as a linear combination of three mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Refer to Fig. 34.21 in which $OP^2 = OM^2 + ON^2$ (Two dimensions)

or $$\vec{r}^{\,2} = x^2 + y^2 \qquad ......(i)$$

and in Fig. 34.22

$$OP^2 = OA^2 + OB^2 + OC^2$$

$$\vec{r}^{\,2} = x^2 + y^2 + z^2 \qquad ....(ii)$$

Magnitude of $\vec{r} = |\vec{r}|$ in case of (i) is $\sqrt{x^2 + y^2}$

and (ii) is $\sqrt{x^2 + y^2 + z^2}$

Note : Given any three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (not necessarily mutually perpendicular unit vectors) any vector $\vec{d}$ is expressible as a linear combination of

$$\vec{a},\ \vec{b} \text{ and } \vec{c}, \text{ i.e., } \vec{d} = x\,\vec{a} + y\,\vec{b} + z\,\vec{c}$$

Example 34.9 A vector of 10 Newton is 30° north of east. Find its components along east and north directions.

Solution : Let $\hat{i}$ and $\hat{j}$ be the unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ (East and North respectively) Resolve OP in the direction OX and OY.

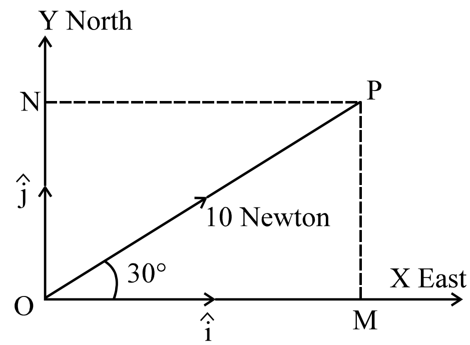


Fig. 34.23

$$\therefore \quad \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON}$$

$$= 10\cos 30°\ \hat{i} + 10\sin 30°\ \hat{j}$$

$$= 10.\frac{\sqrt{3}}{2}\hat{i} + 10.\frac{1}{2}\hat{j}$$

$$= 5\sqrt{3}\ \hat{i} + 5\ \hat{j}$$

$\therefore$ Component along (i) East $= 5\sqrt{3}$ Newton

(ii) North = 5 Newton

Example 34.10 Show that the following vectors are coplanar :

$$\vec{a} - 2\vec{b},\ 3\vec{a} + \vec{b} \text{ and } \vec{a} + 4\vec{b}$$

Solution : The vectors will be coplanar if there exists scalars x and y such that

$$\vec{a} + 4\vec{b} = x(\vec{a} - 2\vec{b}) + y(3\vec{a} + \vec{b})$$

$$= (x + 3y)\vec{a} + (-2x + y)\vec{b} \qquad .....(i)$$

Comparing the co-efficients of $\vec{a}$ and $\vec{b}$ on both sides of (i), we get

$$x + 3y = 1 \text{ and } -2x + y = 4$$

which on solving, gives $x = -\frac{11}{7}$ and $y = \frac{6}{7}$

As $\vec{a} + 4\vec{b}$ is expressible in terms of $\vec{a} - 2\vec{b}$ and $3\vec{a} + \vec{b}$, hence the three vectors are coplanar.

Example 34.11 Given $\vec{r}_1 = \hat{i} - \hat{j} + \hat{k}$ and $\vec{r}_2 = 2\hat{i} - 4\hat{j} - 3\hat{k}$, find the magnitudes of

(a) $\vec{r}_1$ (b) $\vec{r}_2$ (c) $\vec{r}_1 + \vec{r}_2$ (d) $\vec{r}_1 - \vec{r}_2$

Solution :

(a) $|\vec{r}_1| = |\hat{i} - \hat{j} + \hat{k}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

(b) $|\vec{r_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

(c) $\vec{r_1} + \vec{r_2} = (\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 4\hat{j} - 3\hat{k}) = 3\hat{i} - 5\hat{j} - 2\hat{k}$

$\therefore \quad |\vec{r_1} + \vec{r_2}| = |3\hat{i} - 5\hat{j} - 2\hat{k}| = \sqrt{3^2 + (-5)^2 + (-2)^2} = \sqrt{38}$

(d) $\vec{r_1} - \vec{r_2} = (\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - 4\hat{j} - 3\hat{k}) = -\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \quad |\vec{r_1} - \vec{r_2}| = |-\hat{i} + 3\hat{j} + 4\hat{k}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{26}$

Example 34.12 Determine the unit vector parallel to the resultant of two vectors

$$\vec{a} = 3\hat{i} + 2\hat{j} - 4\hat{k} \text{ and } \vec{b} = \hat{i} + \hat{j} + 2\hat{k}$$

Solution : The resultant vector $\vec{R} = \vec{a} + \vec{b} = (3\hat{i} + 2\hat{j} - 4\hat{k}) + (\hat{i} + \hat{j} + 2\hat{k})$

$$= 4\hat{i} + 3\hat{j} - 2\hat{k}$$

Magnitude of the resultant vector $\vec{R}$ is $|\vec{R}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

$\therefore$ The unit vector parallel to the resultant vector

$$\frac{\vec{R}}{|\vec{R}|} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

Example 34.13 Find a unit vector in the direction of $\vec{r} - \vec{s}$

where $\vec{r} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{s} = 2\hat{i} - \hat{j} + 2\hat{k}$

Solution : $\vec{r} - \vec{s} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + 2\hat{k})$

$$= -\hat{i} + 3\hat{j} - 5\hat{k}$$

$\therefore \quad |\vec{r} - \vec{s}| = \sqrt{(-1)^2 + (3)^2 + (-5)^2} = \sqrt{35}$

$\therefore$ Unit vector in the direction of $(\vec{r} - \vec{s})$

$$= \frac{1}{\sqrt{35}}(-\hat{i} + 3\hat{j} - 5\hat{k}) = -\frac{1}{\sqrt{35}}\hat{i} + \frac{3}{\sqrt{35}}\hat{j} - \frac{5}{\sqrt{35}}\hat{k}$$

Example 34.14 Find a unit vector in the direction of $2\vec{a} + 3\vec{b}$ where $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} - \hat{k}$.

Solution : $2\vec{a} + 3\vec{b} = 2(\hat{i} + 3\hat{j} + \hat{k}) + 3(3\hat{i} - 2\hat{j} - \hat{k})$

$$= (2\hat{i} + 6\hat{j} + 2\hat{k}) + (9\hat{i} - 6\hat{j} - 3\hat{k})$$

$$= 11\hat{i} - \hat{k}.$$

$\therefore \qquad |2\vec{a}+3\vec{b}| = \sqrt{(11)^2+(-1)^2} = \sqrt{122}$

$\therefore$ Unit vector in the direction of $(2\vec{a}+3\vec{b})$ is $\frac{11}{\sqrt{122}}\hat{i} - \frac{1}{\sqrt{122}}\hat{k}$.

Example 34.15 Show that the following vectors are coplanar :

$4\vec{a}-2\vec{b}-2\vec{c}$, $-2\vec{a}+4\vec{b}-2\vec{c}$ and $-2\vec{a}-2\vec{b}+4\vec{c}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

Solution : If these vectors be co-planar, it will be possible to express one of them as a linear combination of other two.

Let $\qquad -2\vec{a}-2\vec{b}+4\vec{c} = x(4\vec{a}-2\vec{b}-2\vec{c}) + y(-2\vec{a}+4\vec{b}-2\vec{c})$

where x and y are scalars,

Comparing the co-efficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ from both sides, we get

$$4x-2y=-2, \; -2x+4y=-2 \text{ and } -2x-2y=4$$

These three equations are satisfied by $x=-1, \; y=-1$ Thus,

$$-2\vec{a}-2\vec{b}+4\vec{c} = (-1)(4\vec{a}-2\vec{b}-2\vec{c}) + (-1)(-2\vec{a}+4\vec{b}-2\vec{c})$$

Hence the three given vectors are co-planar.

CHECK YOUR PROGRESS 34.4

1. Write the condition that $\vec{a}, \vec{b}$ and $\vec{c}$ are co-planar.

2. Determine the resultant vector $\vec{r}$ whose components along two rectangular Cartesian co-ordinate axes are 3 and 4 units respectively.

3. In the adjoining figure :

   $|OA| = 4, \; |OB| = 3$ and

   $|OC| = 5$. Express OP in terms of its component vectors.

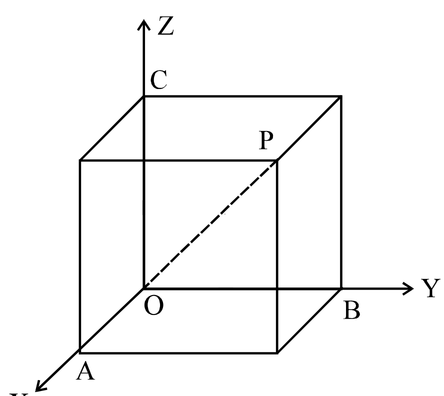


Fig. 34.24

4. If $\vec{r_1} = 4\hat{i}+\hat{j}-4\hat{k}$, $\vec{r_2} = -2\hat{i}+2\hat{j}+3\hat{k}$ and

   $\vec{r_3} = \hat{i}+3\hat{j}-\hat{k}$ then show that

   $|\vec{r_1}+\vec{r_2}+\vec{r_3}| = 7$.

5. Determine the unit vector parallel to the resultant of vectors :

   $\vec{a} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$

6. Find a unit vector in the direction of vector $3\vec{a} - 2\vec{b}$ where $\vec{a} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

7. Show that the following vectors are co-planar :

   $3\vec{a} - 7\vec{b} - 4\vec{c}$, $3\vec{a} - 2\vec{b} + \vec{c}$ and $\vec{a} + \vec{b} + 2\vec{c}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

34.10 SECTION FORMULA

Recall that the position vector of a point P is space with respect to an origin of reference O is $\vec{r} = \overrightarrow{OP}$.

In the following, we try to find the position vector of a point dividing a line segment joining two points in the ratio m : n internally.

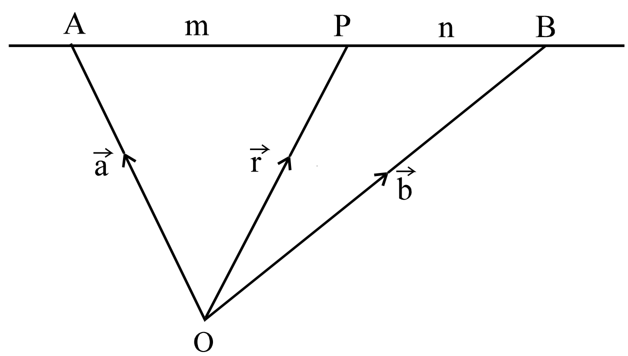


Fig. 34.25

Let A and B be two points and $\vec{a}$ and $\vec{b}$ be their position vectors w.r.t. the origin of reference O, so that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Let P divide AB in the ratio m : n so that

$$\frac{AP}{PB} = \frac{m}{n} \quad \text{or,} \quad n\overrightarrow{AP} = m\overrightarrow{PB} \qquad \text{....(i)}$$

Since $\quad n\overrightarrow{AP} = m\overrightarrow{PB}$, it follows that

$$n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$$

or $\quad (m+n)\overrightarrow{OP} = m\overrightarrow{OB} + n\overrightarrow{OA}$

or $\quad \overrightarrow{OP} = \dfrac{m\overrightarrow{OB} + n\overrightarrow{OA}}{m+n}$

or $\quad \vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

where $\vec{r}$ is the position vector of P with respect to O.

Corollary 1 : If $\frac{m}{n} = 1 \Rightarrow$ m = n, then P becomes mid-point of AB.

$\therefore$ The position vector of the mid-point of the join of two given points, whose position vectors are $\vec{a}$ and $\vec{b}$, is given by $\frac{1}{2}(\vec{a} + \vec{b})$.

Corollary 2 : The position vector P can also be written as

$$\vec{r} = \frac{\vec{a} + \frac{m}{n}\vec{b}}{1 + \frac{m}{n}} = \frac{\vec{a} + k\,\vec{b}}{1 + k}, \qquad .....(ii)$$

where $\qquad k = \frac{m}{n},\ k \neq -1$.

(ii) represents the position vector of a point which divides the join of two points with position vectors $\vec{a}$ and $\vec{b}$, in the ratio k : 1.

Corollary 3 : The position vector of a point P which divides AB in the ratio m : n externally is

$$\vec{r} = \frac{n\,\vec{a} - m\,\vec{b}}{n - m}$$ [**Hint :** This division is in the ratio –m : n]

Example 34.16 Find the position vector of a point which divides the join of two points whose position vectors are given by $\vec{x}$ and $\vec{y}$ in the ratio 2 : 3 internally.

Solution : Let $\vec{r}$ be the position vector of the point.

$$\therefore \qquad \vec{r} = \frac{3\,\vec{x} + 2\,\vec{y}}{3 + 2} = \frac{1}{5}(3\,\vec{x} + 2\,\vec{y}).$$

Example 34.17 Find the position vector of mid-point of the line segment AB, if the position vectors of A and B are respectively, $\vec{x} + 2\,\vec{y}$ and $2\,\vec{x} - \vec{y}$.

Solution : Position vector of mid-point of AB

$$= \frac{(\vec{x} + 2\,\vec{y}) + (2\,\vec{x} - \vec{y})}{2}$$

$$= \frac{3}{2}\vec{x} + \frac{1}{2}\vec{y}$$

Example 34.18 The position vectors of vertices A, B and C of ΔABC are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Find the position vector of the centroid of ΔABC.

Solution : Let D be the mid-point of side BC of ΔABC.

Let G be the centroid of ΔABC. Then G divides AD in the ratio 2 : 1 i.e. AG : GD = 2 : 1.

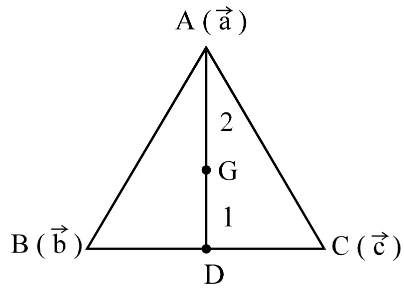


Fig. 34.26

Now position vector of D is $\dfrac{\vec{b}+\vec{c}}{2}$

$\therefore$ Position vector of G is $\dfrac{2\cdot\dfrac{\vec{b}+\vec{c}}{2}+1\cdot\vec{a}}{2+1}$

$$=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

CHECK YOUR PROGRESS 34.5

1. Find the position vector of the point C if it divides AB in the ratio (i) $\dfrac{1}{2} : \dfrac{1}{3}$

   (ii) $2 \;:-3$, given that the position vectors of A and B are $\vec{a}$ and $\vec{b}$ respectively.
2. Find the point which divides the join of $P(\vec{p})$ and $Q(\vec{q})$ internally in the ratio 3 : 4.
3. CD is trisected at points P and Q. Find the position vectors of points of trisection, if the position vectors of C and D are $\vec{c}$ and $\vec{d}$ respectively
4. Using vectors, prove that the medians of a triangle are concurrent.
5. Using vectors, prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

34.11 DIRECTION COSINES OF A VECTOR

In the adjoining figure $\overrightarrow{AB}$ is a vector in the space and $\overrightarrow{OP}$ is the position vector of the point P(x, y, z) such that $\overrightarrow{OP} \parallel \overrightarrow{AB}$. Let $\overrightarrow{OP}$ makes angles α, β and γ respectively with the positive directions of x, y and z axis respectively. α, β and γ are called direction angles of vector $\overrightarrow{OP}$ and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called its direction cosines.

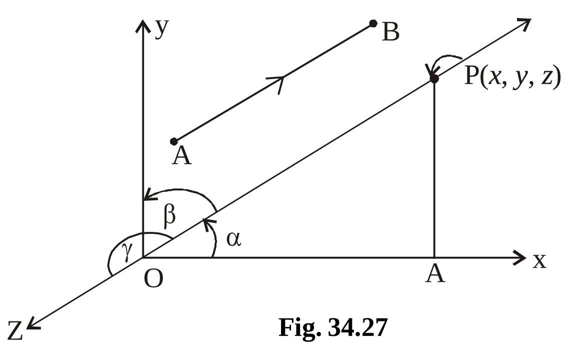


Fig. 34.27

Since $\overrightarrow{OP} \parallel \overrightarrow{AB}$, therefore $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are direction cosines of vector $\overrightarrow{AB}$ also.

Direction cosines of a vector are the cosines of the angles subtended by the vector with the positive directions of *x, y* and *z* axes respectively.

- By reversing the direction, we observe that $\overrightarrow{PO}$ makes angles $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$ with the positive directions of *x, y* and *z* axes respectively. So $\cos(\pi - \alpha) = -\cos\alpha$, $\cos(\pi - \beta) = -\cos\beta$ and $\cos(\pi - \gamma) = -\cos\gamma$ are the direction cosines of $\overrightarrow{PO}$. In fact any vector in space can be extended in two directions so it has two sets of direction cosines. If $(\cos\alpha, \cos\beta, \cos\gamma)$ is one set of direction cosines then $(-\cos\alpha, -\cos\beta, -\cos\gamma)$ is the other set. It is enough to mention any one set of direction cosines of a vector.
- Direction cosines of a vector are usually denoted by *l*, *m* and *n*. In other words $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.
- Since $\overrightarrow{OX}$ makes angles 0°, 90° and 90° with $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ respectively. Therefore cos0°, cos90°, cos90° i.e. 1, 0, 0 are the direction cosines of x-axis. Similarly direction cosines of *y* and *z* axes are (0, 1, 0) and (0, 0, 1) respectively.

  In the figure, 1 let $|\overrightarrow{OP}| = r$. and $PA \perp OX$.

  Now in right angled ΔOAP, $\dfrac{OA}{OP} = \cos\alpha$

  i.e. $\quad OA = OP\cos\alpha$

  i.e. $\quad x = r.l \Rightarrow \quad x = l\,r$

  Similarly by dropping perpendiculars to *y* and *z* axes respectively we get $y = mr$ and $z = nr$.

  Now $\quad x^2 + y^2 + z^2 = r^2(l^2 + m^2 + n^2) \qquad \text{...(i)}$

  But $\quad |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

  or $\quad |\overrightarrow{OP}|^2 = x^2 + y^2 + z^2$

  $\quad = r^2$

  therefore from (i) $l^2 + m^2 + n^2 = 1$

  Again $l = \dfrac{x}{r}$, $m = \dfrac{y}{r}$, $n = \dfrac{z}{r}$

  i.e. $\quad l = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$, $m = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$, $n = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

Hence, if P(*x*, *y*, *z*) is a point in the space, then direction cosines of $\overrightarrow{OP}$ are $\dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$, $\dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$, $\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

34.11.1 DIRECTION COSINES OF A VECTOR JOINING TWO POINTS :

In the adjoining figure $\overrightarrow{PQ}$ is a vector joining points P(x_1 y_1 z) and Q(x_2 y_2 z_2). If we shift the origin to the point P($x_1 y_1 z_1$) without changing the direction of coordinate axes. The coordinates of point Q becomes ($x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$) therefore direction cosines of $\overrightarrow{PQ}$ are $\frac{x_2 - x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$, $\frac{y_2 - y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$,

$$\frac{z_2 - z_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}},$$

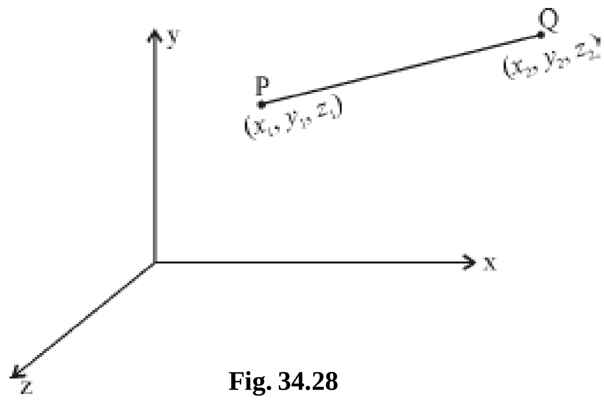


Fig. 34.28

34.11.2 DIRECTION RATIOS OF A VECTOR :

Any three real numbers which are proportional to the direction cosines of a vector are called direction ratios of that vector. Let *l*, *m*, *n* be the direction cosines of a vector and *a*, *b*, *c* be the direction ratios.

then, $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \lambda$ (say)

$\Rightarrow \quad a = \lambda l, b = \lambda m, c = \lambda n$

$\therefore \quad a^2 + b^2 + c^2 = \lambda^2 (l^2 + m^2 + n^2)$

$\Rightarrow \quad \lambda^2 = a^2 + b^2 + c^2$

i.e. $\lambda = \pm\sqrt{a^2 + b^2 + c^2}$

$$\therefore \quad l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{\pm c}{\sqrt{a^2+b^2+c^2}}$$

- If a, b, c are direction ratios of a vector then for every $\lambda \neq 0$, λa, λb, λc are also its direction ratios. Thus a vector can have infinite number of direction ratios.
- If P(x,y,z) is a point in the space, then the direction ratios of $\overrightarrow{OP}$ are x, y, z.
- If P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are two points in the space then the direction ratios of $\overrightarrow{PQ}$ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.
- $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$ in general.

Example 34.19 Let P be a point in space such that OP = $\sqrt{3}$ and $\overrightarrow{OP}$ makes angles $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3}$ with positve directions of, x, y and z axes respectively. Find coordinates of point P.

Solution : d.c.s of $\overrightarrow{OP}$ are $\cos\frac{\pi}{3}, \cos\frac{\pi}{4}, \cos\frac{\pi}{3}$ i.e. $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

$\therefore$ coordinates of point are $x = lr = \frac{1}{2} \times \sqrt{3} = \frac{\sqrt{3}}{2}$

$$y = mr = \frac{1}{\sqrt{2}} \times \sqrt{3} = \frac{\sqrt{3}}{\sqrt{2}}$$

and $$z = nr = \frac{1}{2} \times \sqrt{3} = \frac{\sqrt{3}}{2}$$

Example 34.20 If P(1, 2, –3) is a point in the space, find the direction cosines of vector $\overrightarrow{OP}$.

Solution :

$$l = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{\sqrt{14}}$$

$$m = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{2}{\sqrt{14}}$$

$$n = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{-3}{\sqrt{14}}$$

Example 34.21 Can $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ be direction cosines of a vector.

Solution : $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{4}{3} \neq 1$

$\therefore$ $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ can not be direction cosines of a vector.

Example 34.22 If P(2, 3, –6) and Q(3, –4, 5) are two points in the space. Find the direction cosines of $\overrightarrow{OP}, \overrightarrow{QO}$ and $\overrightarrow{PQ}$, where O is the origin.

Solution : D.C.'S of $\overrightarrow{OP}$ are $\dfrac{2}{\sqrt{2^2+3^2+(-6)^2}}, \dfrac{3}{\sqrt{2^2+3^2+(-6)^2}}, \dfrac{-6}{\sqrt{2^2+3^2+(-6)^2}}$

i.e. $\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{-6}{7}$.

Similarly d.c.'s of $\overrightarrow{QO}$ are $\dfrac{-3}{5\sqrt{2}}, \dfrac{4}{5\sqrt{2}}, \dfrac{-5}{\sqrt{2}}$

D.C.'s of $\overrightarrow{PQ}$ are : $\dfrac{3-2}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}, \dfrac{-4-3}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}$

$\dfrac{5+6}{\sqrt{(3-2)^2+(-4-3)^2+(5+6)^2}}$

i.e. $\dfrac{1}{\sqrt{171}}, \dfrac{-7}{\sqrt{171}}, \dfrac{11}{\sqrt{171}}$

Example 34.23 Find the direction cosines of a vector which makes equal angles with the axes.

Solution : Suppose the given vector makes angle α with each of the $\overrightarrow{OX}, \overrightarrow{OY}$ and $\overrightarrow{OZ}$. Therefore $\cos\alpha$, $\cos\alpha$, $\cos\alpha$ are the direction cosines of the vector.

Now, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

i.e. $\cos\alpha = \pm\dfrac{1}{\sqrt{3}}$

$\therefore$ d.c.'s of the vector are $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

Example 34.24 If P(1, 2, –3) and Q(4, 3, 5) are two points in space, find the direction ratios of $\overrightarrow{OP}, \overrightarrow{QO}$ and $\overrightarrow{PQ}$

Solution : d.r.'s of $\overrightarrow{OP}$ are 1, 2, –3

d.r.'s of $\overrightarrow{QO}$ are (–4, –3, –5) or (4, 3, 5)

d.r.'s of $\overrightarrow{PQ}$ are 4 –1, 3 –2, 5 – (–3)

i.e. 3, 1, 8.

CHECK YOUR PROGRESS 34.6

1. Fill in the blanks:
   (i) Direction cosines of y-axis are...
   (ii) If l, m, n are direction cosines of a vector, then $l^2 + m^2 + n^2 = ...$

(iii) If a, b, c are direction ratios of a vector, then $a^2 + b^2 + c^2$ is .... to 1

(iv) The direction cosines of a vector which makes equal angles with the coordinate axes are...

(v) If two vectors are parallel to each other then their direction ratios are...

(vi) (1, –1, 1) are not direction cosines of any vector because...

(vii) The number of direction ratios of a vector are... (finite/infinite)

2. If P(3, 4, –5) is a point in the space. Find the direction cosines of $\overrightarrow{OP}$.

3. Find the direction cosines of $\overrightarrow{AB}$ where A(–2, 4, –5) and B(1, 2, 3) are two points in the space.

4. If a vector makes angles 90°, 135° and 45° with the positive directions of *x*, *y* and *z* axis respectively, find its direction ratios.

34.12 PRODUCT OF VECTORS

In Section 34.9, you have multiplied a vector by a scalar. The product of vector with a scalar gives us a vector quantity. In this section we shall take the case when a vector is multiplied by another vector. There are two cases :

(i) When the product of two vectors is a scalar, we call it a scalar product, also known as dot product corresponding to the symbol '$\bullet$' used for this product.

(ii) When the product of two vectors is a vector, we call it a vector product, also known as cross product corresponding to the symbol '$\times$' used for this product.

34.13 SCALAR PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ two vectors and θ be the angle between them. The scalar product, denoted by $\vec{a}$, $\vec{b}$, is defined by

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Clearly, $\vec{a}\cdot\vec{b}$ is a scalar as $|\vec{a}|$, $|\vec{b}|$ and $\cos\theta$ are all scalars.

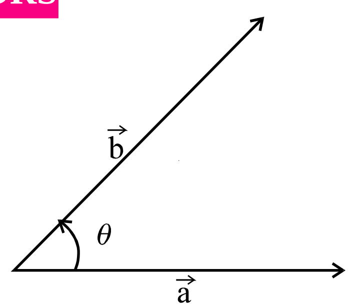


Fig. 34.29

Remarks

1. If $\vec{a}$ and $\vec{b}$ are like vectors, then $\vec{a}\cdot\vec{b} = ab\cos\theta = ab$, where a and b are magnitudes of $\vec{a}$ and $\vec{b}$.

2. If $\vec{a}$ and $\vec{b}$ are unlike vectors, then $\vec{a}\cdot\vec{b} = ab\cos\pi = -ab$

4. Angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

5. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ and $\vec{a}\cdot(\vec{b}+\vec{c}) = (\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c})$.

6. $n(\vec{a}\cdot\vec{b}) = (n\vec{a})\cdot\vec{b} = \vec{a}\cdot(n\vec{b})$ where n is any real number.

7. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ as $\hat{i}$, $\hat{j}$ and $\hat{k}$ are mutually perpendicular unit vectors.

Example 34.25 If $\vec{a} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, find $\vec{a} \cdot \vec{b}$.

Also find angle between $\vec{a}$ and $\vec{b}$.

Solution : $\vec{a} \cdot \vec{b} = (3\hat{i} + 2\hat{j} - 6\hat{k}) \cdot (4\hat{i} - 3\hat{j} + \hat{k})$

$$= 3 \times 4 + 2 \times (-3) + (-6) \times 1$$

$$\left[\because \ \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \text{ and } \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0\right]$$

$$= 12 - 6 - 6 = 0$$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$

Then $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = 0$

$\therefore$ $\theta = \dfrac{\pi}{2}$.

34.14 VECTOR PRODUCT OF TWO VECTORS

Before we define vector product of two vectors, we discuss below right handed and left handed screw and associate it with corresponding vector triad.

34.14.1 Right Handed Screw

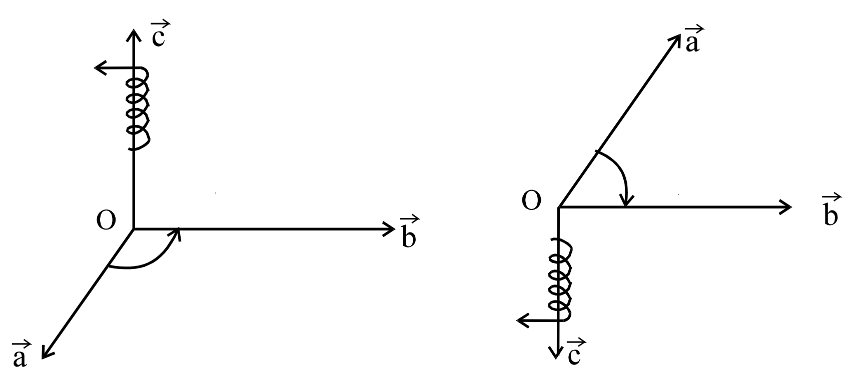
If a screw is taken and rotated in the anticlockwise direction, it translates towards the reader. It is called ***right handed screw***.

34.14.2 Left handed Screw

If a screw is taken and rotated in the clockwise direction, it translates away from the reader. It is called a left handed screw.

Now we associate a screw with given ordered vector triad.

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors whose initial point is O.



(i) (ii)

Fig. 34.30

Now if a right handed screw at O is rotated from $\vec{a}$ towards $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 34.28 (i)]

Similarly if a left handed screw at O is rotated from $\vec{a}$ to $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 34.28 (ii)]. This time the direction of translation will be opposite to the first one.

Thus an ordered vector triad $\vec{a}, \vec{b}, \vec{c}$ is said to be right handed or left handed according as the right handed screw translated along $\vec{c}$ or opposite to $\vec{c}$ when it is rotated through an angle less than 180°.

34.14.3 VECTOR (CROSS) PRODUCT OF THE VECTORS :

If $\vec{a}$ and $\vec{b}$ are two non zero vectors then their cross product is denoted by $\vec{a} \times \vec{b}$ and defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta . \hat{n}$

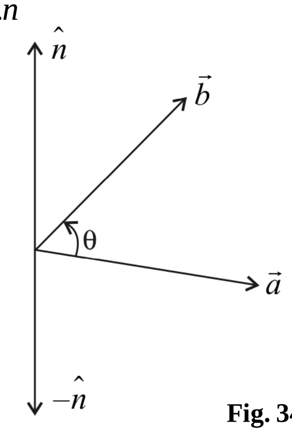


Fig. 34.31

Where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$; such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system (see figure) i.e. the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{n}$.

- $\vec{a} \times \vec{b}$ is a vector and $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
- If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then θ is not defined and in that case we consider $\vec{a} \times \vec{b} = \vec{0}$.
- If $\vec{a}$ and $\vec{b}$ are non zero vectors. Then $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are collinear or parallel vectors. i.e. $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \| \vec{b}$.
- In particular $\vec{b} \times \vec{b} = \vec{0}$ and $\vec{b} \times (-\vec{b}) = \vec{0}$ because in the first situation $\theta = 0$ and in 2nd case $\theta = \pi$. Making the value of $\sin\theta = 0$ in both the cases.
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$
- $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$ and $\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$.

- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$
- $\lambda(\vec{a} \times \vec{b}) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b})$.
- Angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given as

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}.$$

- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given by $\frac{1}{2}|\vec{a} \times \vec{b}|$.
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a} \times \vec{b}|$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\hat{a} \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

Example 34.26 Using cross product find the angle between the vectors $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 3 & -2 & 1 \end{vmatrix} = \hat{i}(1-6) - \hat{j}(2+9) + \hat{k}(-4-3)$$

$$= -5\hat{i} - 11\hat{j} - 7\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{25+121+49} = \sqrt{195}$$

$$|\vec{a}| = \sqrt{4+1+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$$

$$\therefore \quad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{195}}{\sqrt{14}.\sqrt{14}} = \frac{\sqrt{195}}{14}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{\sqrt{195}}{14}\right)$$

Example 34.27 Find a unit vector perpendicular to each of the vectors $\vec{a} = 3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -3 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-2+3) - \hat{j}(-3+3) + \hat{k}(3-2)$$

$$\vec{a} \times \vec{b} = \hat{i} + \hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{1+1} = \sqrt{2}$$

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ $= \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$= \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

Example 34.28 Find the area of the triangle having point A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

Solution :

$$\overrightarrow{AB} = (1-1)\hat{i} + (2-1)\hat{j} + (3-1)\hat{k}$$

$$= \hat{j} + 2\hat{k}$$

$$\overrightarrow{AC} = (2-1)\hat{i} + (3-1)\hat{j} + (1-1)\hat{k}$$

$$= \hat{i} + 2\hat{j}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = \hat{i}(0-4) - \hat{j}(0-2) + \hat{k}(0-1)$$

$$= -4\hat{i} + 2\hat{j} - \hat{k}$$

$$\therefore \quad |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-4)^2 + (2)^2 + (-1)^2} = \sqrt{16+4+1} = \sqrt{21}$$

Hence, Area of ΔABC $= \dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \dfrac{\sqrt{21}}{2}$ unit^2.

Example 34.29 Find the area of the parallelogram having A(5, –1, 1), B(–1, –3, 4), C(1, –6, 10) and D(7, –4, 7) as its vertices.

Solution :

$$\overrightarrow{AB} = (-1-5)\hat{i}+(-3+1)\hat{j}+(4-1)\hat{k}$$

$$= -6\hat{i}-2\hat{j}+3\hat{k}$$

$$\overrightarrow{AD} = (7-5)\hat{i}+(-4+1)\hat{j}+(7-1)\hat{k}$$

$$= 2\hat{i}-3\hat{j}+6\hat{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & -2 & 3 \\ 2 & -3 & 6 \end{vmatrix} = \hat{i}(-12+9)-\hat{j}(-36-6)+\hat{k}(18+4)$$

$$= -3\hat{i}+42\hat{j}+22\hat{k}$$

$$|\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{9+1764+484} = \sqrt{2257} \text{ unit}^2.$$

CHECK YOUR PROGRESS 34.7

1. (i) If $\vec{a}\times\vec{b}$ is a unit vector and $|\vec{a}|=3, |\vec{b}|=\frac{\sqrt{2}}{3}$, then the angle between $\vec{a}$ and $\vec{b}$ is ...

   (ii) If $|\vec{a}.\vec{b}| = |\vec{a}\times\vec{b}|$, then angle between $\vec{a}$ and $\vec{b}$ is ...

   (iii) The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ is ...

2. Find a unit vector perpendicular to both the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.

3. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a}=3\hat{i}+\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$.

4. If $\vec{a}=2\hat{i}+2\hat{j}+2\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$, $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+j\vec{b}$ is perpendicular to $\vec{c}$, find the value of j.

34.15 SCALAR TRIPLE PRODUCT :

If $\vec{a},\vec{b}$ and $\vec{c}$ are any three vectors then the scalar product of $\vec{a}\times\vec{b}$ with $\vec{c}$ is called scalar triple product i.e. $(\vec{a}\times\vec{b}).\vec{c}$ is called scalar triple product of $\vec{a},\vec{b}$ and $\vec{c}$. It is usually denoted as $[\vec{a}.\vec{b}\vec{c}]$

- $[\vec{a}\,\vec{b}\,\vec{c}]$ is a scalar quantity.
- $(\vec{a}\times\vec{b}).\vec{c}$ represents the volume of a parallelopiped having $\vec{a},\vec{b},\vec{c}$ as coterminous edges.
- $(\vec{a}\times\vec{b}).\vec{c}=0$ if $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar vectors or any two of the three vectors are equal or parallel.
- In the scalar triple product the position of dot and cross can be interchanged provided the cyclic order of the vectors is maintained i.e.

$$(\vec{a}\times\vec{b}).\vec{c} = \vec{a}.(\vec{b}\times\vec{c})$$

$$(\vec{b}\times\vec{c}).\vec{a} = \vec{b}.(\vec{c}\times\vec{a})$$

$$(\vec{c}\times\vec{a}).\vec{b} = \vec{c}.(\vec{a}\times\vec{b})$$

- $(\vec{a}\times\vec{b}).\vec{c}=(\vec{b}\times\vec{c}).\vec{a}=(\vec{c}\times\vec{a}).\vec{b}$
- $(\vec{a}\times\vec{b}).\vec{c}=-(\vec{b}\times\vec{a}).\vec{c}=-\vec{c}.(\vec{b}\times\vec{a})$

  $(\vec{b}\times\vec{c}).\vec{a}=-(\vec{c}\times\vec{b}).\vec{a}=-\vec{a}.(\vec{c}\times\vec{b})$

  $(\vec{c}\times\vec{a}).\vec{b}=-(\vec{a}\times\vec{c}).\vec{b}=-\vec{b}.(\vec{a}\times\vec{c})$
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}, \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

  then $(\vec{a}\times\vec{b}).\vec{c}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- Four points A, B, C and D are coplanar if $\overrightarrow{AB},\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar i.e. $(\overrightarrow{AB}\times\overrightarrow{AC}).\overrightarrow{AD}=0$

Example 34.30 Find the volume of the parallelepiped whose edges are represented by $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.

Solution : Volume $=(\vec{a}\times\vec{b}).\vec{c} = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$$= 2(4-1)+3(2+3)+4(-1-6)$$

$$= 6+15-28=-7$$

Neglecting negative sign, required volume = 7 unit^3.

Example 34.31 Find the value of λ so that the vectors

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k},\ b = \hat{i} + 2\hat{j} - 3\hat{k},\ c = 3\hat{i} + \lambda\hat{j} + 5\hat{k} \text{ are coplanar.}$$

Solution : The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ will be coplanar if $[\vec{a}\vec{b}\vec{c}] = 0$

i.e. $$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

i.e. $2(10 + 3\lambda) + 1(5 + 9) + 1(\lambda - 6) = 0$

i.e. $7\lambda + 28 = 0$

$\Rightarrow$ $\lambda = -4.$

Example 34.32 Show that the four points A, B, C and D whose position vectors are $(4\hat{i} + 5\hat{j} + \hat{k}), (-\hat{j} - \hat{k}),\ (3\hat{i} + 9\hat{j} + 4\hat{k})$ and $(-4\hat{i} + 4\hat{j} + 4\hat{k})$ respectively are coplanar.

Solution :
$$\overrightarrow{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k}$$
$$\overrightarrow{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}$$
$$\overrightarrow{AD} = -8\hat{i} - \hat{j} + 3\hat{k}$$

Now $$(\overrightarrow{AB} \times \overrightarrow{AC}) . \overrightarrow{AD} = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12 + 3) + 6(-3 + 24) - 2(1 + 32)$$

$$= -60 + 126 - 66 = 0$$

Hence, A, B, C and D are coplanar.

Example 34.33 Prove that $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$

Solution :

$$\begin{aligned}
\text{LHS} &= (\vec{a} + \vec{b}) . [(\vec{b} + \vec{c}) \times \vec{c} + \vec{a})] \\
&= (\vec{a} + \vec{b}) . [(\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}] \\
&= (\vec{a} + \vec{b}) . [(\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}] \qquad \because\ \vec{c} \times \vec{c} = 0 \\
&= \vec{a} . (\vec{b} \times \vec{c}) + \vec{a} . (\vec{b} \times \vec{a}) + \vec{a} . (\vec{c} \times \vec{a}) \\
&\quad + \vec{b} . (\vec{b} \times \vec{c}) + \vec{b} . (\vec{b} \times \vec{a}) + \vec{b} . (\vec{c} \times \vec{a}) \\
&= \vec{a} . (\vec{b} \times \vec{c}) + \vec{b} . (\vec{c} \times \vec{a}) \qquad [\because \text{ scalar triple product is zero when two vector are equal}] \\
&= 2[\vec{a}\,\vec{b}\,\vec{c}] \\
&= \text{RHS}
\end{aligned}$$

CHECK YOUR PROGRESS 34.8

1. Find the volume of the parallelopiped whose edges are represented by $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{c} = 3\hat{i} + 2\hat{j} + 5\hat{k}$.

2. Find the value of λ so that the vectors $\vec{a} = -4\hat{i} - 6\hat{j} + \lambda\hat{k}, \vec{b} = -\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{c} = -8\hat{i} - \hat{j} + 3\hat{k}$ are coplanar.

LET US SUM UP

- A physical quantity which can be represented by a number only is called a scalar.
- A quantity which has both magnitude and direction is called a vector.
- A vector whose magnitude is 'a' and direction from A to B can be represented by $\overrightarrow{AB}$ and its magnitude is denoted by $|\overrightarrow{AB}| = a$.
- A vector whose magnitude is equal to the magnitude of another vector $\vec{a}$ but of opposite direction is called negative of the given vector and is denoted by $-\vec{a}$.
- A unit vector is of magnitude unity. Thus, a unit vector parallel to $\vec{a}$ is denoted by $\hat{a}$ and is equal to $\dfrac{\vec{a}}{|\vec{a}|}$.
- A zero vector, denoted by $\vec{0}$, is of magnitude 0 while it has no definite direction.
- Unlike addition of scalars, vectors are added in accordance with triangle law of addition of vectors and therefore, the magnitude of sum of two vectors is always less than or equal to sum of their magnitudes.
- Two or more vectors are said to be collinear if their supports are the same or parallel.
- Three or more vectors are said to be coplanar if their supports are parallel to the same plane or lie on the same plane.
- If $\vec{a}$ is a vector and x is a scalar, then $x\vec{a}$ is a vector whose magnitude is $|x|$ times the magnitude of $\vec{a}$ and whose direction is the same or opposite to that of $\vec{a}$ depending upon $x > 0$ or $x < 0$.
- Any vector co-planar with two given non-collinear vectors is expressible as their linear combination.
- Any vector in space is expressible as a linear combination of three given non-coplanar vectors.
- The position vector of a point that divides the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio of m : n internally/externally are given by
- $\dfrac{n\vec{a} + m\vec{b}}{m+n}, \dfrac{n\vec{a} - m\vec{b}}{n-m}$ respectively.

- The position vector of mid-point of the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\frac{\vec{a}+\vec{b}}{2}$$

- The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

- The vector product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between $\vec{a}$, $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.
- Direction cosines of a vector are the cosines of the angles subtended by the vector with the positive directions of *x*, *y* and *z* axes respectively.
- Any three real numbers which are proportional to the direction cosines of a vector are called direction ratios of that vector.
- Usually, direction cosines of a vector are denoted by *l*, *m*, *n* and direction ratios by *a*, *b*, *c*.
- $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$, in general.
- If $\overrightarrow{AB} = x\hat{i} + y\hat{j} + z\hat{k}$, then direction ratios of $\overrightarrow{AB}$ are x, y, z and direction cosines are

$$\frac{\pm x}{\sqrt{x^2+y^2+z^2}}, \frac{\pm y}{\sqrt{x^2+y^2+z^2}}, \frac{\pm z}{\sqrt{x^2+y^2+z^2}}.$$

- Direction cosines of a vector are unique but direction ratios are infinite.
- Cross product of two non zero vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta.\hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.
- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$.
- $\vec{a}\times\vec{b} = \vec{0}$ if either $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a}$ and $\vec{b}$ are parallel or $\vec{a}$ and $\vec{b}$ are collinear.
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$
- $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \vec{j}$.
- $\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\vec{j}$.
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
- $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$

- $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

- Area of $\Delta = \dfrac{1}{2}|\vec{a}\times\vec{b}|$ where $\vec{a}$ and $\vec{b}$ represent adjacent sides of a triangle.

- Area of $\|$ gm $= |\vec{a}\times\vec{b}|$ where $\vec{a}$ and $\vec{b}$ represent adjacent sides of the parallelogram.

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors then $(\vec{a}\times\vec{b}).\vec{c}$ is called scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$. It is usually denoted as $[\vec{a}\,\vec{b}\,\vec{c}]$

- Volume of parallelepiped $= (\vec{a}\times\vec{b}).\vec{c}$ where $\vec{a}, \vec{b}, \vec{c}$ represent coterminous edges of the parallelopiped.

- $(\vec{a}\times\vec{b}).\vec{c} = 0$, if $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar or any two of the three vectors are equal or parallel.

- $(\vec{a}\times\vec{b}).\vec{c} = \vec{a}.(\vec{b}\times\vec{c})$

- $(\vec{a}\times\vec{b}).\vec{c} = (\vec{b}\times\vec{c}).\vec{a} = (\vec{c}\times\vec{a}).\vec{b}$

- $(\vec{a}\times\vec{b}).\vec{c} = -(\vec{b}\times\vec{a}).\vec{c}$

- Four points A, B, C and D are coplanar if $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar i.e. $(\overrightarrow{AB}\times\overrightarrow{AC}).\overrightarrow{AD} = 0$.

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$(\vec{a}\times\vec{b}).\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

SUPPORTIVE WEB SITES

www.youtube.com/watch?v=ihNZlp7iUHE

http://emweb.unl.edu/math/mathweb/vectors/vectors.html

http://www.mathtutor.ac.uk/geometry_vectors

www.khanacademy.org/.../introduction-to-**vectors-and-scalars**

TERMINAL EXERCISE

1. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that any two of them are non-collinear. Find their sum if the vector $\vec{a}+\vec{b}$ is collinear with the vector $\vec{c}$ and if the vector $\vec{b}+\vec{c}$ is collinear with $\vec{a}$.

2. Prove that any two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exist numbers x and y, both not zero simultaneously, such that $x\vec{a}+y\vec{b}=\vec{0}$.

3. ABCD is a parallelogram in which M is the mid-point of side CD. Express the vectors $\overrightarrow{BD}$ and $\overrightarrow{AM}$ in terms of vectors $\overrightarrow{BM}$ and $\overrightarrow{MC}$.

4. Can the length of the vector $\vec{a}-\vec{b}$ be (i) less than, (ii) equal to or (iii) larger than the sum of the lengths of vectors $\vec{a}$ and $\vec{b}$?

5. Let $\vec{a}$ and $\vec{b}$ be two non-collinear vectors. Find the number x and y, if the vector $(2-x)\vec{a}+\vec{b}$ and $y\vec{a}+(x-3)\vec{b}$ are equal.

6. The vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find the number x if the vector $3\vec{a}+x\vec{b}$ and $(1-x)\vec{a}-\frac{2}{3}\vec{b}$ are parallel.

7. Determine x and y such that the vector $\vec{a}=-2\hat{i}+3\hat{j}+y\hat{k}$ is collinear with the vector $\vec{b}=x\hat{i}-6\hat{j}+2\hat{k}$. Find also the magnitudes of $\vec{a}$ and $\vec{b}$.

8. Determine the magnitudes of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ if $\vec{a}=3\hat{i}-5\hat{j}+8\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-4\hat{k}$.

9. Find a unit vector in the direction of $\vec{a}$ where $\vec{a}=-6\hat{i}+3\hat{j}-2\hat{k}$.

10. Find a unit vector parallel to the resultant of vectors $3\hat{i}-2\hat{j}+\hat{k}$ and $-2\hat{i}+4\hat{j}+\hat{k}$

11. The following forces act on a particle P :

    $\vec{F}_1=2\hat{i}+\hat{j}-3\hat{k}$, $\vec{F}_2=-3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{F}_3=3\hat{i}-2\hat{j}+\hat{k}$ measured in Newtons.

    Find (a) the resultant of the forces, (b) the magnitude of the resultant.

12. Show that the following vectors are co-planar :

$(\vec{a}-2\vec{b}+\vec{c})$, $(2\vec{a}+\vec{b}-3\vec{c})$ and $(-3\vec{a}+\vec{b}+2\vec{c})$

where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three non-coplanar vectors.

13. A vector makes angles $\frac{\pi}{3}, \frac{\pi}{3}$ with $\overline{OX}$ and $\overline{OY}$ respectively. Find the angle made by it with $\overline{OZ}$.

14. If $P(\sqrt{3}, 1, 2\sqrt{3})$ is a point in space, find direction cosines of $\overline{OP}$ where O is the origin.

15. Find the direction cosines of the vector joining the points $(-4, 1, 7)$ and $(2, -3, 2)$.

16. Using the concept of direction ratios show that $\overrightarrow{PQ} \parallel \overrightarrow{RS}$ where coordinates of P, Q, R and S are (0, 1, 2), (3, 4, 8), $\left(-2, \frac{3}{2}, -3\right)$ and $\left(\frac{5}{2}, 6, 6\right)$ respectively.

17. If the direction ratios of a vector are (3, 4, 0). Find its directions cosines.

18. Find the area of the parallelogram whose adjacent sides are represented by the vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

19. Find the area of the ΔABC where coordinates of A, B, C are (3, –1, 2), (1, –1, –3) and (4, –3, 1) respectively.

20. Find a unit vector perpendicular to each of the vectors $2\hat{i}-3\hat{j}+\hat{k}$ and $3\hat{i}-4\hat{j}-\hat{k}$.

21. If $\vec{A}=2\hat{i}-3\hat{j}-6\hat{k}$ and $\vec{B}=\hat{i}+4\hat{j}-2\hat{k}$, then find $(\vec{A}+\vec{B})\times(\vec{A}-\vec{B})$.

22. Prove that : $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$.

23. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, show that $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$.

24. Find the volume of the parallelepiped whose edges are represented by $\vec{a}=2\hat{i}-4\hat{j}+5\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}, \vec{c}=3\hat{i}-5\hat{j}+2\hat{k}$.

25. Show that the vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}-3\hat{j}-5\hat{k}$ and $\vec{c}=3\hat{i}-4\hat{j}-4\hat{k}$ are coplanar.

26. Find the value of λ if the points A(3, 2, 1), B(4, λ, 5), C(4, 2, –2) and D(6, 5, –1) are coplanar.

ANSWERS

CHECK YOUR PROGRESS 34.1

1. (d) 2. (b)

3.

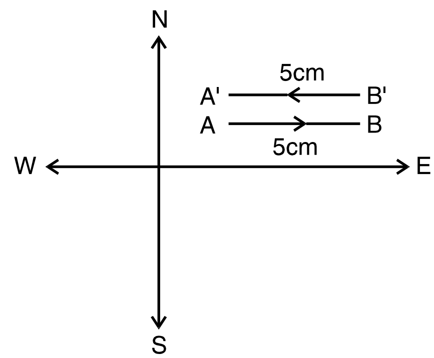


Fig. 34.32

4. Two vectors are said to be like if they have same direction what ever be their magnitudes.

But in case of equal vectors magnitudes and directions both must be same.

5.

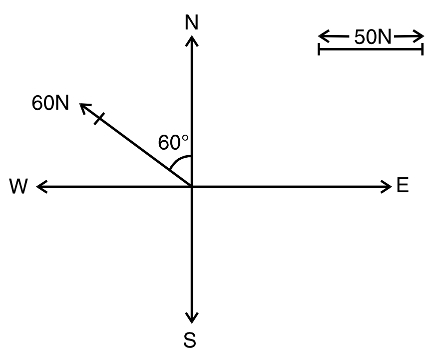


Fig. 34.33

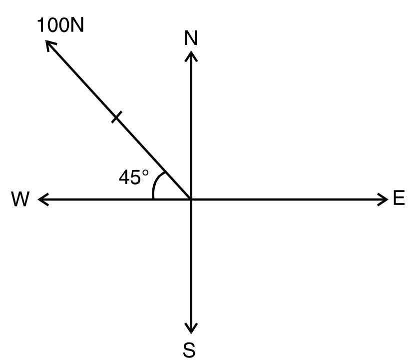


Fig. 34.34

CHECK YOUR PROGRESS 34.2

1. $\vec{0}$ 2. $\vec{0}$

CHECK YOUR PROGRESS 34.3

1. $\vec{b} - \vec{a}$

2. (i) It is a vector in the direction of $\vec{a}$ and whose magnitudes is 3 times that of $\vec{a}$.

(ii) It is a vector in the direction opposite to that of $\vec{b}$ and with magnitude 5 times that of $\vec{b}$.

3. $\overrightarrow{DB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = 2\vec{a} + 3\vec{b}$.

4. $|y\vec{n}| = y|\vec{n}|$ if $y > 0$

5. Vector $= -y|\vec{n}|$ if $y < 0 = 0$ if $y = 0$

6. $\vec{p} = x\vec{q}$, x is a non-zero scalar.

CHECK YOUR PROGRESS 34.4

1. If there exist scalars x and y such that $\vec{c} = x\vec{a} + y\vec{b}$

2. $\vec{r} = 3\hat{i} + 4\hat{j}$ 3. $\overrightarrow{OP} = 4\hat{i} + 3\hat{j} + 5\hat{k}$

5. $\frac{1}{7}\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$ 6. $\frac{1}{\sqrt{51}}\hat{i} - \frac{5}{\sqrt{51}}\hat{j} - \frac{5}{\sqrt{51}}\hat{k}$

CHECK YOUR PROGRESS 34.5

1. (i) $\frac{1}{5}(2\vec{a} + 3\vec{b})$ (ii) $(3\vec{a} - 2\vec{b})$

2. $\frac{1}{7}(4\vec{p} + 3\vec{q})$ 3. $\frac{1}{3}(2\vec{c} + \vec{d}),\ \frac{1}{3}(\vec{c} + 2\vec{d})$

CHECK YOUR PROGRESS 34.6

1. (i) (0, 1, 0)   (ii) 1   (iii) not equal   (iv) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

(v) proportional   (vi) sum of their squares is not equal to 1   (vii) infinite

2. $\frac{3}{5\sqrt{2}}, \frac{2\sqrt{2}}{5}, \frac{-1}{\sqrt{2}}$ 3. $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$ 4. 0, –1, 1

CHECK YOUR PROGRESS 34.7

1. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) 1

2. $\frac{-\hat{i}}{\sqrt{6}} + \frac{2\hat{j}}{\sqrt{6}} - \frac{\hat{k}}{\sqrt{6}}$.   3. $\sqrt{42}$ unit^2

CHECK YOUR PROGRESS 34.8

1. 42 unit^3 2. $\lambda = -2$

TERMINAL EXERCISE

1. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

3. $\overrightarrow{BD} = \overrightarrow{BM} - \overrightarrow{MC}, \ \overrightarrow{AM} = \overrightarrow{BM} + 2\overrightarrow{MC}$

4. (i) Yes, $\vec{a}$ and $\vec{b}$ are either any non-collinear vectors or non-zero vectors of same direction.

   (ii) Yes, $\vec{a}$ and $\vec{b}$ are either in the opposite directions or at least one of them is a zero vector.

   (iii) Yes, $\vec{a}$ and $\vec{b}$ have opposite directions.

5. $x = 4, \quad y = -2$ 6. $x = 2, \quad -1$

7. $x = 4, \quad y = -1$ $|\vec{a}| = \sqrt{14}, \ |\vec{b}| = 2\sqrt{14}$

8. $|\vec{a} + \vec{b}| = 6, \ |\vec{a} - \vec{b}| = 14$

9. $-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{2}{7}\hat{k}$ 10. $\pm\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

11. $2\hat{i} + \hat{j}\,;\sqrt{5}$ 13. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$

14. $\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$ 15. $\frac{6}{\sqrt{77}}, \frac{-4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}$

17. $\frac{3}{5}, \frac{4}{5}, 0$ 18. $8\sqrt{3}$ unit^2

19. $\frac{1}{2}\sqrt{165}$ unit^2 20. $\frac{7\hat{i} + 5\hat{j} + \hat{k}}{\sqrt{75}}$

21. $-60\hat{i} + 4\hat{j} - 22\hat{k}$ 24. 8 unit^3

26. $\lambda = 5$

35

PLANE

Look closely at a room in your house. It has four walls, a roof and a floor. The floor and roof are parts of two parallel planes extending infinitely beyond the boundary. You will also see two pairs of parallel walls which are also parts of parallel planes. Similarly, the tops of tables, doors of rooms etc. are examples of parts of planes.

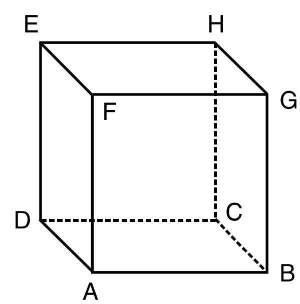


Fig. 35.1

If we consider any two points in a plane, the line joining these points will lie entirely in the same plane. This is the characteristic of a plane.

Look at Fig.35.1.You know that it is a representation of a rectangular box. This has six faces, eight vertices and twelve edges.

The pairs of opposite and parallel faces are

(i) ABCD and FGHE

(ii) AFED and BGHC

(iii) ABGF and DCHE

and the sets of parallel edges are given below :

(i) AB, DC EH and FG

(ii) AD, BC, GH and FE

(iii) AF, BG, CH and DE

Each of the six faces given above forms a part of the plane, and there are three pairs of parallel planes, denoted by the opposite faces.

In this lesson, we shall establish the general equation of a plane, the equation of a plane passing through three given points, the intercept form of the equation of a plane and the normal form of the equation of a plane. We shall show that a homogeneous equation of second degree in three variables x,y and z represents a pair of planes. We shall also find the equation of a plane bisecting the angle between two planes and area of a triangle in space.

OBJECTIVES

After studying this lesson, you will be able to :

- identify a plane;

MODULE - IX
Vectors and three dimensional Geometry

Notes

- establish the equation of a plane in normal form;
- find the general equation of a plane passing through a given point;
- find the equation of a plane passing through three given points;
- find the equation of a plane in the intercept form and normal form;
- find the angle between two given planes;

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of three dimensional geometry.
- Direction cosines and direction ratio of a line.
- Projection of a line segment on another line.
- Condition of perpendicularity and parallelism of two lines in space.

35.1 VECTOR EQUATION OF A PLANE

A plane is uniquely determined if any one of the following is known:

(i) Normal to the plane and its distance from the origin is given.

(ii) One point on the plane is given and normal to the plane is also given.

(iii) It passes through three given non collinear points.

35.2 EQUATION OF PLANE IN NORMAL FROM

Let the distance (OA) of the plane from origin O be d and let $\hat{n}$ be a unit vector normal to the plane. Consider $\vec{r}$ as position vector of an arbitarary point P on the plane.

Since OA is the perpendicular distance of the plane from the origin and $\hat{n}$ is a unit vector perpendicular to the plane.

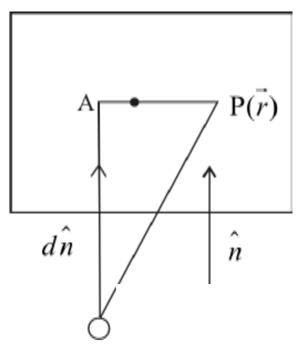


Fig. 35.2

$$\therefore \qquad \overrightarrow{OA} = d\hat{n}$$

$$\text{Now} \qquad \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - d\hat{n}$$

$\overrightarrow{OA}$ is perpendicular to the plane and $\overrightarrow{AP}$ lies in the plane, therefore $\overrightarrow{OA} \perp \overrightarrow{AP}$

$$\Rightarrow \qquad \overrightarrow{AP}\,.\,\overrightarrow{OA} = 0$$

$$\text{i.e.} \qquad (\vec{r} - d\hat{n}).\hat{n} = 0$$

$$\text{i.e.} \qquad \vec{r}.\hat{n} - d = 0$$

$$\text{i.e.} \qquad \vec{r}.\hat{n} = d \qquad \ldots(3)$$

which is the equation of plane in vector from.

35.3 CONVERSION OF VECTOR FORM INTO CARTESIAN FORM

Let (x, y, z) be the co-ordinates of the point P and l, m, n be the direction cosines of $\hat{n}$.

Then $\vec{r} \;=\; x\hat{i}+y\hat{j}+z\hat{k}$

$$\hat{n} \;=\; l\hat{i}+m\hat{j}+n\hat{k}$$

Substituting these value in equation (3) we get

$$(x\hat{i}+y\hat{j}+z\hat{k}).(l\hat{i}+m\hat{y}+n\hat{k}) \;=\; d$$

$$\Rightarrow \qquad lx+my+nz \;=\; d$$

This is the corresponding Cartesian form of equation of plane in normal form.

Note : In equation (3), if $\vec{r}.\vec{n}=d$ is the equation of the plane then d is not the distance of the plane from origin. To find the distance of the plane from origin we have to convert $\vec{n}$ into $\hat{n}$ by dividing both sides by $|\vec{n}|$. Therefore $\dfrac{d}{|\vec{n}|}$ is the distance of the plane from the origin.

Example 15.1 Find the distance of the plane $\vec{r}.(6\hat{i}-3\hat{j}-2\hat{k})-1=0$ from the origin. Also find the direction cosines of the unit vector perpendicular to the plane.

Solution : The given equation can be written as

$$\vec{r}.(6\hat{i}-3\hat{j}-2\hat{k}) \;=\; 1$$

$$|6\hat{i}-3\hat{j}-2\hat{k}| \;=\; \sqrt{36+9+4}=7$$

Dividing both sides of given equation by 7 we get

$$\frac{\vec{r}.(6\hat{i}-3\hat{j}-2\hat{k})}{7} \;=\; \frac{1}{7}$$

i.e. $$\vec{r}.\left(\frac{6}{7}\hat{i}-\frac{3}{7}\hat{j}-\frac{2}{7}\hat{k}\right) \;=\; \frac{1}{7}$$

$\therefore$ d.c.'s of unit vector normal to the plane are $\dfrac{6}{7},\dfrac{-3}{7},\dfrac{-2}{7}$ and distance of plane from origin $=\dfrac{1}{7}$

35.4 EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN VECTOR

Let $\vec{a}$ be the position vector of the given point A and $\vec{r}$ the position vector of an arbitrary point on the plane. $\vec{n}$ is a vector perpendicular to the plane.

Now $\quad \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= \vec{r} - \vec{a}$

Now $\quad \vec{n} \perp (\vec{r} - \vec{a})$

$\therefore \quad (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad ...(4)$

This is vector equation of plane in general form.

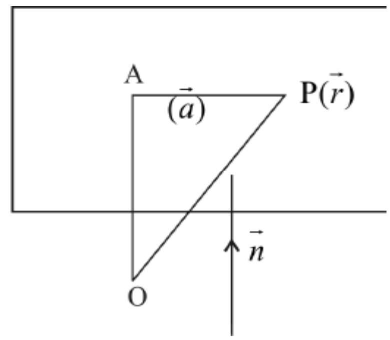


Fig. 35.3

35.5 CARTESIAN FORM

Let (x_1, y_1, z_1) be the coordinates of the given point A and (x, y, z) be the coordinates of point P. Again let a, b, c be the direction ratios of normal vector $\vec{n}$.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$$

Substituting these values in equation (4) we get

$$\{(x - x_1)i + (y - y_1)j + (z - z_1)k\}.\{ai + bj + ck\} = 0$$

$$\Rightarrow \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

which is the corresponding Cartesian form of the equation of plane.

Example 35.2 Find the vector equation of a plane passing through the point (5, 5, –4) and perpendicular to the line with direction ratios 2, 3, –1.

Solution : Here $\quad \vec{a} = 5\hat{i} + 5\hat{j} - 4\hat{k}$

and $\quad \vec{n} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\therefore$ Equation of plane is $\left(\vec{r} - (5\hat{i} + 5\hat{j} - 4\hat{k})\right).(2\hat{i} + 3\hat{j} - \hat{k}) = 0$

35.6 EQUATION OF A PLANE PASSING THROUGH THREE NON COLLINEAR POINTS

(a) Vector Form

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the given points Q, R and S respectively.

Let $\vec{r}$ be the position vector of an arbitrary point P on the plane.

Vectors $\overrightarrow{QR} = \vec{b} - \vec{a}, \overrightarrow{QS} = \vec{c} - \vec{a}$ and $\overrightarrow{QP} = \vec{r} - \vec{a}$ lie in the same plane and $\overrightarrow{QR} \times \overrightarrow{QS}$ is a vector perpendicular to both $\overrightarrow{QR}$ and $\overrightarrow{QS}$. Therefore $\overrightarrow{QR} \times \overrightarrow{QS}$ is perpendicular to $\overrightarrow{QP}$ also.

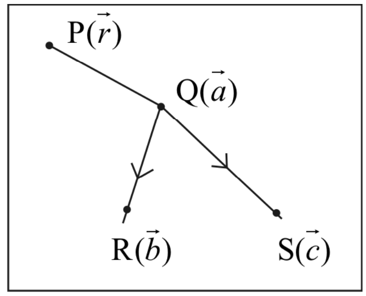


Fig. 35.4

$$\therefore \qquad \overrightarrow{QP} \cdot (\overrightarrow{QR} \times \overrightarrow{QS}) = 0$$

$$(\vec{r}-\vec{a}).\left\{(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})\right\} = 0 \qquad \ldots(5)$$

This is the equation of plane in vector form.

(b) Cartesian Form

Let (x, y, z), (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of the points P, Q, R and S respectively.

$$\therefore \quad \overrightarrow{QP} = \vec{r} - \vec{a} = (x-x_1)\hat{i} + (y-y_1)\hat{j} + (z-z_1)\hat{k}$$

$$\overrightarrow{QR} = \vec{b} - \vec{a} = (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}$$

$$\overrightarrow{QS} = \vec{c} - \vec{a} = (x_3-x_1)\hat{i} + (y_3-y_1)\hat{j} + (z_3-z_1)\hat{k}$$

Substituting these values in equation (5) we get.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

which is the equation of plane in Cartesian form.

Example 35.3 Find the vector equation of the plane passing through the points Q(2, 5, –3), R(–2, –3, 5) and S(5, 3, –3).

Solution : Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of points Q, R and S respectively and $\vec{r}$ be the position vector of an arbitrary point on the plane.

Vector equation of plane is $\{\vec{r}-\vec{a}\}.\left\{(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})\right\} = 0$

Here
$$\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$$
$$\vec{b} = -2\hat{i} - 3\hat{j} + 5\hat{k}$$
$$\vec{c} = 5\hat{i} + 3\hat{j} - 3\hat{k}$$

$$\vec{b}-\vec{a} = -4\hat{i}-8\hat{j}+8\hat{k}$$
$$\vec{c}-\vec{a} = 3\hat{i}-2\hat{j}$$

$\therefore$ Required equation is $\{\vec{r}-(2\hat{i}+5\hat{j}-3\hat{k})\}.\{(-4\hat{i}-8\hat{j}+8\hat{k})\times(3\hat{i}-2\hat{j})\} = 0$

35.7 EQUATION OF A PLANE IN THE INTERCEPT FORM

Let a, b,c be the lengths of the intercepts made by the plane on the x,y and z axes respectively. It implies that the plane passes through the points (a,0,0), (0,b,0) and (0,0,c)

Putting $x_1 = a \quad y_1 = 0 \quad z_1 = 0$

$x_2 = 0 \quad y_2 = b \quad z_2 = 0$

and $x_3 = 0 \quad y_3 = 0 \quad z_3 = c$ in (A),

we get the required equation of the plane as

$$\begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

which on expanding gives $bcx + acy + abz - abc = 0$

or $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \qquad .....(B)$$

Equation (B) is called the ***Intercept form*** of the equation of the plane.

Example 35.4 Find the equation of the plane passing through the points (0,2,3), (2,0,3) and (2,3,0).

Solution : Using (A), we can write the equation of the plane as

$$\begin{vmatrix} x-0 & y-2 & z-3 \\ 2-0 & 0-2 & 3-3 \\ 2-0 & 3-2 & 0-3 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & y-2 & z-3 \\ 2 & -2 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0$$

or $\quad x(6-0)-(y-2)(-6)+(z-3)(2+4) = 0$

or $\quad 6x+6(y-2)+6(z-3) = 0$

or $\quad x+y-2+z-3 = 0 \quad$ or $\quad x+y+z = 5$

Example 35.5 Show that the equation of the plane passing through the points (2,2,0), (2,0,2) and (4,3,1) is x = y + z.

Solution : Equation of the plane passing through the point (2,2,0) is

$$a(x-2)+b(y-2)+cz=0 \qquad .....(i)$$

$\because$ (i) passes through the point (2, 0, 2)

$\therefore \qquad a(2-2)+b(0-2)+2c=0$

or $\qquad c=b \qquad .....(ii)$

Again (i) passes through the point (4, 3, 1)

$\therefore \qquad a(4-2)+b(3-2)+c=0$

or $\qquad 2a+b+c=0 \qquad ....(iii)$

From (ii) and (iii), we get $2a+2b=0$ or $a=-b$

$\therefore$ (i) becomes

$$-b(x-2)+b(y-2)+bz=0$$

or $\qquad -(x-2)+y-2+z=0$

or $\qquad y+z-x=0$

or $\qquad x=y+z$

Example 35.6 Reduce the equation of the plane $4x-5y+6z-60=0$ to the intercept form. Find its intercepts on the co-ordinate axes.

Solution : The equation of the plane is

$$4x-5y+6z-60=0 \quad \text{or} \quad 4x-5y+6z=60 \qquad ....(i)$$

The equation (i) can be written as $\dfrac{4x}{60}-\dfrac{5y}{60}+\dfrac{6z}{60}=1$ or $\dfrac{x}{15}+\dfrac{y}{(-12)}+\dfrac{z}{10}=1$

which is the interecept form of the equation of the plane and the intercepts on the co-ordinate axes are $15, -12$ and 10 respectively.

Example 35.7 Reduce each of the following equations of the plane to the normal form :

(i) $2x-3y+4z-5=0$ (ii) $2x+6y-3z+5=0$

Find the length of perpendicular from origin upon the plane in both the cases.

Solution : (i) The equation of the plane is $2x-3y+4z-5=0 \qquad ....(A)$

Dividing (A) by $\sqrt{2^2+\left(-3^2\right)+4^2}$ or , by $\sqrt{29}$

we get, $\qquad \dfrac{2x}{\sqrt{29}}-\dfrac{3y}{\sqrt{29}}+\dfrac{4z}{\sqrt{29}}-\dfrac{5}{\sqrt{29}}=0$

or $\qquad \dfrac{2x}{\sqrt{29}}-\dfrac{3y}{\sqrt{29}}+\dfrac{4z}{\sqrt{29}}=\dfrac{5}{\sqrt{29}}$

which is the equation of the plane in the normal form.

$\therefore$ Length of the perpendicular is $\dfrac{5}{\sqrt{29}}$

(ii) The equation of the plane is $2x+6y-3z+5=0 \qquad .....(B)$

Dividing (B) by $\sqrt{2^2 + 6^2 + (-3)^2}$

or by -7 we get, [refer to corollary 2]

$$-\frac{2x}{7} - \frac{6y}{7} + \frac{3z}{7} - \frac{5}{7} = 0 \quad \text{or} \quad -\frac{2x}{7} - \frac{6y}{7} + \frac{3z}{7} = \frac{5}{7}$$

which is the required equation of the plane in the normal form.

$\therefore$ Length of the perpendicular from the origin upon the plane is $\frac{5}{7}$

Example 35.8 The foot of the perpendicular drawn from the origin to the plane is $(4, -2, -5)$. Find the equation of the plane.

Solution : Let P be the foot of perpendicular drawn from origin O to the plane.

Then P is the point $(4, -2, -5)$.

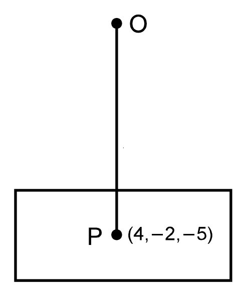


Fig. 35.5

The equation of a plane through the point $P(4, -2, -5)$ is

$$a(x-4) + b(y+2) + c(z+5) = 0 \qquad .....(i)$$

Now OP $\perp$ plane and direction cosines of OP are proportional to

$$4-0, -2-0, -5-0$$

i.e. $\qquad 4, -2, -5$.

Substituting $4, -2$ and -5 for a, b and c in (i), we get

$$4(x-4) - 2(y+2) - 5(z+5) = 0$$

or $\qquad 4x - 16 - 2y - 4 - 5z - 25 = 0$

or $\qquad 4x - 2y - 5z = 45$

which is the required equation of the plane.

CHECK YOUR PROGRESS 35.1

1. Reduce each of the following equations of the plane to the normal form :
   (i) $4x + 12y - 6z - 28 = 0$
   (ii) $3y + 4z + 3 = 0$
2. The foot of the perpendicular drawn from the origin to a plane is the point $(1, -3, 1)$. Find the equation of the plane.
3. The foot of the perpendicular drawn from the origin to a plane is the point $(1, -2, 1)$. Find the equation of the plane.
4. Find the equation of the plane passing through the points
   (a) $(2, 2, -1)$, $(3, 4, 2)$ and $(7, 0, 6)$

(b) $(2, 3, -3)$, $(1, 1, -2)$ and $(-1, 1, 4)$

(c) $(2, 2, 2)$, $(3, 1, 1)$ and $(6, -4, -6)$

5. Show that the equation of the plane passing through the points $(3, 3, 1)$, $(-3, 2-1)$ and $(8, 6, 3)$ is $4x + 2y - 13z = 5$

6. Find the equation of a plane whose intercepts on the coordinate axes are 2,3 and 4 respectively.

7. Find the intercepts made by the plane $2x + 3y + 4z = 24$ on the co-ordinate axes.

8. Show that the points $(-1, 4, -3)$, $(3, 2, -5)$, $(-3, 8, -5)$ and $(-3, 2, 1)$ are coplanar.

9. (i) What are the direction cosines of a normal to the plane $x - 4y + 3z = 7$.?

   (ii) What is the distance of the plane $2x + 3y - z = 17$ from the origin?

   (iii) The planes $\vec{r}.(\hat{i} - \hat{j} + 3\hat{k}) = 7$ and $\vec{r}.(3\hat{i} - 12\hat{j} - 5\hat{k}) = 6$ are ... to each other.

10. Convert the following equation of a plane in Cartesian form : $\vec{r}.(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1.$

11. Find the vector equation of a plane passing through the point (1, 1, 0), (1, 2, 1) and (–2, 2, –1).

12. Find the vector equation of a plane passing through the point (1, 4, 6) and normal to the vector $\hat{i} - 2\hat{j} + \hat{k}$.

35.6 ANGLE BETWEEN TWO PLANES

Let the two planes p_1 and p_2 be given by

$a_1x + b_1y + c_1z + d_1 = 0$ ....(i)

and $a_2x + b_2y + c_2z + d_2 = 0$ ....(ii)

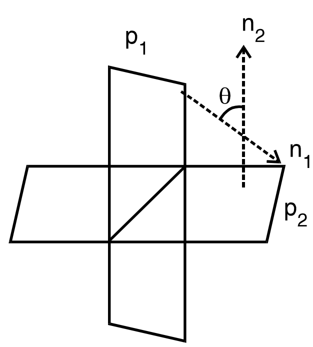


Fig. 35.5

Let the two planes intersect in the line l and let n_1 and n_2 be normals to the two planes. Let θ be the angle between two planes.

$\therefore$ The direction cosines of normals to the two planes are

$$\pm\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \pm\frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \pm\frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

and $$\pm\frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \pm\frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \pm\frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}},$$

$\therefore$ $\cos\theta$ is given by $\cos\theta = \pm\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$

where the sign is so chosen that $\cos\ \theta$ is positive

Corollary 1 :

When the two planes are perpendicular to each other then $\theta = 90°$ i.e., $\cos\theta = 0$

$\therefore$ The condition for two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ to be perpendicular to each other is

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Corollary 2 :

If the two planes are parallel, then the normals to the two planes are also parallel

$$\therefore \qquad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$\therefore$ The condition of parallelism of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and

$$a_2x + b_2y + c_2z + d_2 = 0 \text{ is } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

This implies that the equations of two parallel planes differ only by a constant. Therefore, any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is a constant.

Example 35.9 Find the angle between the planes

$$3x + 2y - 6z + 7 = 0 \qquad \text{......(i)}$$

and $$2x + 3y + 2z - 5 = 0 \qquad \text{......(ii)}$$

Solution : Here $a_1 = 3, b_1 = 2, c_1 = -6$

and $a_2 = 2, b_2 = 3, c_2 = 2$

$\therefore$ If θ is the angle between the planes (i) and (ii), then

$$\cos\theta = \frac{3.2 + 2.3 + (-6).2}{\sqrt{3^2 + 2^2 + (-6)^2}\ \sqrt{2^2 + 3^2 + 2^2}} = 0$$

$\therefore\ \theta = 90°$

Thus the two planes given by (i) and (ii) are perpendicular to each other.

Example 35.10 Find the equation of the plane parallel to the plane $x - 3y + 4z - 1 = 0$ and passing through the point $(3, 1, -2)$.

Solution : Let the equation of the plane parallel to the plane

9. (i) What are the direction cosines of a normal to the plane $x - 4y + 3z = 7$.?
   (ii) What is the distance of the plane $2x + 3y - z = 17$ from the origin?

(iii) The planes $\vec{r}.(\hat{i}-\hat{j}+3\hat{k})=7$ and $\vec{r}.(3\hat{i}-12\hat{j}-5\hat{k})$ = 6 are ... to each other.

10. Convert the following equation of a plane in Cartesian form : $\vec{r}.(2\hat{i}+3\hat{j}-4\hat{k})=1.$

11. Find the vector equation of a plane passing through the point (1, 1, 0), (1, 2, 1) and (–2, 2, –1).

12. Find the vector equation of a plane passing through the point (1, 4, 6) and normal to the vector $\hat{i}-2\hat{j}+\hat{k}$.

$$x - 3y + 4z - 1 = 0 \text{ be } x - 3y + 4z + k = 0 \qquad \text{..... (i)}$$

Since (i) passes through the point $(3,1,-2)$, it should satisfy it

$\therefore \qquad 3-3-8+k=0 \quad \text{or} \quad k=8$

$\therefore$ The required equation of the plane is $x - 3y + 4z + 8 = 0$

Example 35.11 Find the equation of the plane passing through the points $(-1,2,3)$ and $(2,-3,4)$ and which is perpendicular to the plane $3x + y - z + 5 = 0$

Solution : The equation of any plane passing through the point $(-1,2,3)$ is

$$a(x+1)+b(y-2)+c(z-3)=0 \qquad \text{.....(i)}$$

Since the point $(2,-3,4)$ lies on the plane (i)

$\therefore \qquad 3a - 5b + c = 0 \qquad$ ....(ii)

Again the plane (i) is perpendicular to the plane $3x + y - z + 5 = 0$

$\therefore \qquad 3a + b - c = 0 \qquad$ ....(iii)

From (ii) and (iii), by cross multiplication method, we get,

$$\frac{a}{4}=\frac{b}{6}=\frac{c}{18} \quad \text{or} \quad \frac{a}{2}=\frac{b}{3}=\frac{c}{9}$$

Hence the required equation of the plane is

$$2(x+1)+3(y-2)+9(z-3)=0 \quad \text{....[From (i)]}$$

or
$$2x + 3y + 9z = 31$$

Example 35.12 Find the equation of the plane passing through the point $(2,-1,5)$ and perpendicular to each of the planes

$$x + 2y - z = 1 \qquad \text{and } 3x - 4y + z = 5$$

Solution : Equation of a plane passing through the point $(2,-1,5)$ is

$$a(x-2)+b(y+1)+c(z-5)=0 \qquad \text{.....(i)}$$

As this plane is perpendicular to each of the planes

$$x + 2y - z = 1 \qquad \text{and} \qquad 3x - 4y + z = 5$$

We have $\qquad a.1 + b.2 + c.(-1) = 0$

and $\qquad a.3 + b.(-4) + c.(1) = 0$

or $$a + 2b - c = 0 \qquad ....(ii)$$

$$3a - 4b + c = 0 \qquad ....(iii)$$

From (ii) and (iii), we get

$$\frac{a}{2-4} = \frac{b}{-3-1} = \frac{c}{-4-6}$$

or $$\frac{a}{-2} = \frac{b}{-4} = \frac{c}{-10} \quad \text{or} \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{5} = \lambda \text{ (say)}$$

$\therefore$ $$a = \lambda,\ b = 2\lambda \text{ and } c = 5\lambda$$

Substituting for a, b and c in (i), we get

$$\lambda(x-2) + 2\lambda(y+1) + 5\lambda(z-5) = 0$$

or $$x - 2 + 2y + 2 + 5z - 25 = 0$$

or $$x + 2y + 5z - 25 = 0$$

which is the required equation of the plane.

CHECK YOUR PROGRESS 35.2

1. Find the angle between the planes

   (i) $2x - y + z = 6$ and $x + y + 2z = 3$

   (ii) $3x - 2y + z + 17 = 0$ and $4x + 3y - 6z + 25 = 0$

2. Prove that the following planes are perpendicular to each other.

   (i) $x + 2y + 2z = 0$ and $2x + y - 2z = 0$

   (ii) $3x + 4y - 5z = 9$ and $2x + 6y + 6z = 7$

3. Find the equation of the plane passing through the point $(2, 3, -1)$ and parallel to the plane $2x + 3y + 6z + 7 = 0$

4. Find the equation of the plane through the points $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$

5. Find the equation of the plane which passes through the origin and is perpendicular to each of the planes $x + 2y + 2z = 0$ and $2x + y - 2z = 0$

35.9 DISTANCE OF A POINT FROM A PLANE

Let the equation of the plane in normal form be

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p \text{ where } p>0 \qquad ....(i)$$

Case I : Let the point $P(x', y', z')$ lie on the same side of the plane in which the origin lies.

Let us draw a plane through point P parallel to plane (i).Its equation is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p' \qquad .....(ii)$$

where p' is the length of the perpendicular drawn from origin upon the plane given by (ii). Hence the perpendicular distance of P from plane (i) is $p - p'$

As the plane (ii) passes through the point (x', y', z'),

$$x'\cos\alpha + y'\cos\beta + z'\cos\gamma = p'$$

$\therefore$ The distance of P from the given plane is

$$p - p' = p - (x'\cos\alpha + y'\cos\beta + z'\cos\gamma)$$

Case II : If the point P lies on the other side of the plane in which the origin lies, then the distance of P from the plane (i) is,

$$p' - p = x'\cos\alpha + y'\cos\beta + z'\cos\gamma - p$$

Note : If the equation of the plane be given as $ax + by + cz + d = 0$, we have to first convert it into the normal form, as discussed before, and then use the above formula.

Example 35.13 Find the distance of the point (1,2,3) from the plane $3x - 2y + 5z + 17 = 0$

Solution : Required distance $= \dfrac{3.1 - 2.2 + 5.3 + 17}{\sqrt{3^2 + (-2)^2 + 5^2}} = \dfrac{31}{\sqrt{38}}$ units.

Example 35.14 Find the distance between the planes

$$x - 2y + 3z - 6 = 0$$

and $$2x - 4y + 6z + 17 = 0$$

Solution : The equations of the planes are

$$x - 2y + 3z - 6 = 0 \qquad \text{....(i)}$$

$$2x - 4y + 6z + 17 = 0 \qquad \text{....(ii)}$$

Here $$\frac{1}{2} = \frac{(-2)}{(-4)} = \frac{3}{6}$$

$\therefore$ Planes (i) and (ii) are parallel

Any point on plane (i) is (6, 0, 0)

$\therefore$ Distance between planes (i) and (ii) = Distance of point (6,0,0) from (ii)

$$= \frac{2 \times 6 - 4.0 + 6.0 + 17}{\sqrt{(2)^2 + (-4)^2 + 6^2}}$$

$$= \frac{29}{\sqrt{56}} \text{ units } = \frac{29}{2\sqrt{14}} \text{ units}$$

CHECK YOUR PROGRESS 35.3

1. Find the distance of the point

   (i) $(2, -3, 1)$ from the plane $5x - 2y + 3z + 11 = 0$

   (ii) $(3, 4, -5)$ from the plane $2x - 3y + 3z + 27 = 0$

2. Find the distance between planes

$3x + y - z - 7 = 0$ and $6x + 2y - 2z + 11 = 0$

LET US SUM UP

- A plane is a surface such that if any two points are taken on it, the line joining these two points lies wholly in the plane.
- $\vec{r} . \hat{n} = d$ is the vector equation of a plane where $\hat{n}$ is a unit vector normal to the plane and d is the distance of the plane from origin.
- Corresponding cartesian form of the equation is l x +my +n z =d, where *l*,*m*,*n* are the direction cosines of the normal vector to the plane and d is the distance of the plane from origin.
- $\left(\vec{r} - \vec{a}\right) . \vec{n} = 0$ is another vecter equation of a plane where $\vec{a}$ is position vecter of a given point on the plane and $\vec{n}$ is a vecter normal to the plane.
- Corresponding cartesion form of this equation is $a(x-x_1) + b(y-y_1) + c(z-z_1)=0$, where a,b,c are the direction ratios of normal to the plane and (x_1,y_1,z_1) are coordinates of given point on plane.
- $\left(\vec{r} - \vec{a}\right) . \{\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\} = 0$ is the equation of a plane possing through three points with position vecter $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.
- Its corresponding cartesian equation is:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Equation of a plane in the intercept from is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

  where a,b and c are intercepts made by the plane on x,y and z axes respectively.
- Angle θ between two planes $a_1x + b_1y + c_1z + d_1 = 0$

  and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- Two planes are perpendicular to each other if and only if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

- Two planes are parallel if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- Distance of a point (x', y', z') from a plane

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p \text{ is}$$

$\left| p - (x'\cos\alpha + y'\cos\beta + z'\cos\gamma) \right|$, where the point (x', y', z') lies on the same side of the plane in which the origin lies.

SUPPORTIVE WEB SITES

http://www.mathopenref.com/plane.html

http://en.wikipedia.org/wiki/Plane_(geometry)

TERMINAL EXERCISE

1. Find the equation of a plane passing through the point $(-2, 5, 4)$
2. Find the equation of a plane which divides the line segment joining the points $(2, 1, 4)$ and $(2, 6, 4)$ internally in the ratio of $2 : 3$.
3. Find the equation of the plane through the points $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.
4. Show that the four points $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar. Also find the equation of the palne in which they lie.
5. The foot of the perpendicular drawn from $(1, -2, -3)$ to a plane is $(3, 2, -1)$. Find the equation of the plane.
6. Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$
7. Prove that the planes $3x - 5y + 8z - 2 = 0$ and $12x - 20y + 32z + 9 = 0$ are parallel.
8. Determine the value of k for which the planes $3x - 2y + kz - 1 = 0$ and $x + ky + 5z + 2 = 0$ may be perpendicular to each other.
9. Find the distance of the point $(3, 2, -5)$ from the plane $2x - 3y - 5z = 7$
10. Find the vector equation of a plane possing through the point (3,-1,5) and perpendicular to the line with direction ratios (2,-3,1).
11. Find the vector equation of a plane perpendicular to the vecter $3\hat{i} + 5\hat{j} - 6\hat{k}$ and at a distance of 7 units from origin.
12. Find the vector equation of a plane passing through the points A(-2,6,-6), B(-3,10,-9), and C(-5,0,-6).

Notes

ANSWERS

CHECK YOUR PROGRESS 35.1

1. (i) $\frac{4x}{14}+\frac{12y}{14}-\frac{6z}{14}=2$ (ii) $-\frac{3}{5}y-\frac{4}{5}z=\frac{3}{5}$

2. $x-3y+z-11=0$ 3. $x-2y+z-6=0$ 4. (a) $5x+2y-3z-17=0$

(b) $3x-y+2=0$ (c) $x+2y-2=4$ 6. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$

7. Intercepts on x,y & z axes are 12,8,6 respectively.

9. (i) $\frac{1}{\sqrt{26}},\frac{-4}{\sqrt{26}},\frac{3}{\sqrt{26}}$ (ii) $\frac{17}{\sqrt{14}}$ units (iii) perpendicular

10. $2x+3y-4z=1$ 11. $\vec{x}.\left(2\hat{x}+3\hat{y}-3\hat{k}\right)=5$ 12. $\vec{x}.\left(\hat{x}-2\hat{y}+\hat{k}\right)+1=0$

CHECK YOUR PROGRESS 35.2

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{2}$ 3. $2x+3y+6z=7$ 4. $2x+2y-3z+3=0$

5. $2x-2y+z=0$

CHECK YOUR PROGRESS 35.3

1. (i) $\frac{30}{\sqrt{38}}$ units (ii) $\frac{6}{\sqrt{22}}$ units. 2. $\frac{25}{2\sqrt{11}}$ units.

TERMINAL EXERCISE

1. $a(x+2)+b(y-5)+c(z-4)=0$

2. $a(x-2)+b(y-3)+c(z-4)=0$

3. $2x+3y-3z-5=0$ 4. $5x-7y+11z+4=0$ 5. $x+2y+z=6$

6. $\frac{\pi}{3}$ 8. $k=-1$ 9. $\frac{18}{\sqrt{38}}$ 10. $\{\vec{r}-\left(-3\hat{i}+\hat{j}+5\hat{k}\right)\}.\left(2\hat{i}-3\hat{j}+\hat{k}\right)=0$

11. $\vec{r}.\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\}=7$

12. $\{\vec{x}.\left(-2\hat{i}+6\hat{j}-6\hat{k}\right)\}.\{\left(-\hat{i}+4\hat{j}-3\hat{k}\right)\times\left(-3\hat{i}-6\hat{j}\right)\}=0$

36

STRAIGHT LINES

In Fig. 36.1, we see a rectangular box having six faces, which are parts of six planes. In the figure, ABCD and EFGH are parallel planes. Similarly, ADGH and BCFE are parallel planes and so are ABEH and CFGD. Two planes ABCD and CFGD intersect in the line CD. Similarly, it happens with any two adjacent planes. Also two edges, say AB and AH meet in the vertex A. It also happens with any two adjacent edges. We can see that the planes meet in lines and the edges meet in vertices.

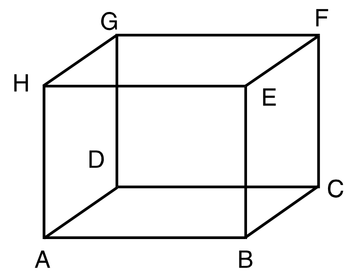


Fig. 36.1

In this lesson, we will study the equations of a line in space in symmetric form, reducing the general equation of a line into symmetric form, finding the perpendicular distance of a point from a line and finding the angle between a line and a plane. We will also establish the condition of coplanarity of two lines.

OBJECTIVES

After studying this lesson, you will be able to :

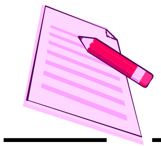
- find the equations of a line in space in symmetric form;
- convert the general equations of a line into symmetric form;
- find the perpendicular distance of a point from a line;
- find the angle between a line and a plane; and
- find the condition of coplanarity of two lines.

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of three dimensional geometry.
- Direction cosines/ratios of a line and projection of a line segment on another line.
- Condition of parallelism and perpendicularity of two lines.
- General equation of a plane.
- Equations of a plane in different forms.
- Angle between two planes.

36.1 VECTOR EQUATION OF A LINE

A line is uniquely defermined if, it passes through a given point and it has a given direction or it passes through two given points.



16.1.1 Equation of a line through a given point and parallel to a given vector : Let l be the line which passes through the point A and which is parallel to the vector $\vec{b}$. Let $\vec{a}$ be position vector of the point A and $\vec{r}$ be the position vector of an arbitrary point P on the line.

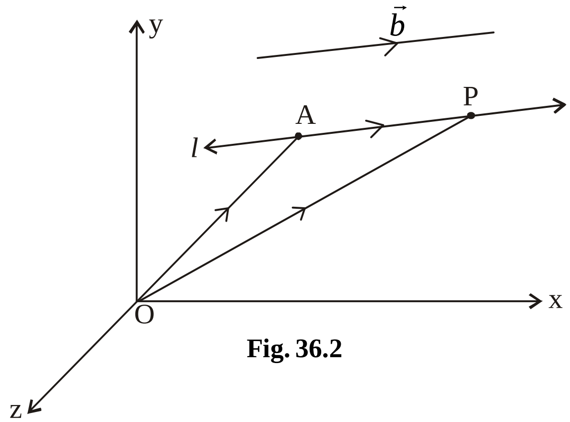


Fig. 36.2

In ΔOAP, $\qquad \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OP}$

i.e. $\qquad \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$

But $\qquad \overrightarrow{AP} \parallel \vec{b} \quad \therefore \quad \overrightarrow{AP} = \lambda \vec{b}$

$\therefore \qquad \vec{r} - \vec{a} = \lambda \vec{b} \qquad$ ...(1)

$\Rightarrow \quad \vec{r} = \vec{a} + \lambda \vec{b}$ is the required equation of the line in vector from

36.1.2 Conversion of Vector form into Cartesian form :

Let (x_1, y_1, z_1) be the coordinates of the given point A and b_1, b_2, b_3 be the direction ratios of vector $\vec{b}$. Consider (x, y, z) as the coordinates of point P.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and $\qquad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}.$

Substituting these values in equation (1) we get

$$(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$\Rightarrow \quad \frac{x - x_1}{b_1} = \lambda, \frac{y - y_1}{b_2} = \lambda, \frac{z - z_1}{b_3} = \lambda$$

$\therefore \quad \dfrac{x - x_1}{b_1} = \dfrac{y - y_1}{b_2} = \dfrac{z - z_1}{b_3}$ is the corresponding Cartesian form of equation of the line. This is also known as symmetric form of equation of line.

36.1.3 EQUATION OF THE LINE PASSING THROUGH TWO GIVEN POINTS :

Let l be the line which passes through two points A and B. Let $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B respectively. Let $\vec{r}$ be the position vector of an arbitary point P on the line.

From the figure,

$$\overrightarrow{AP} = \vec{r} - \vec{a}$$

and $$\overrightarrow{AB} = \vec{b} - \vec{a}$$

But $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear vectors

$$\therefore \quad \overrightarrow{AP} = \lambda \overrightarrow{AB}$$

i.e. $$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a}) \qquad \text{...(2)}$$

$$\Rightarrow \quad \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

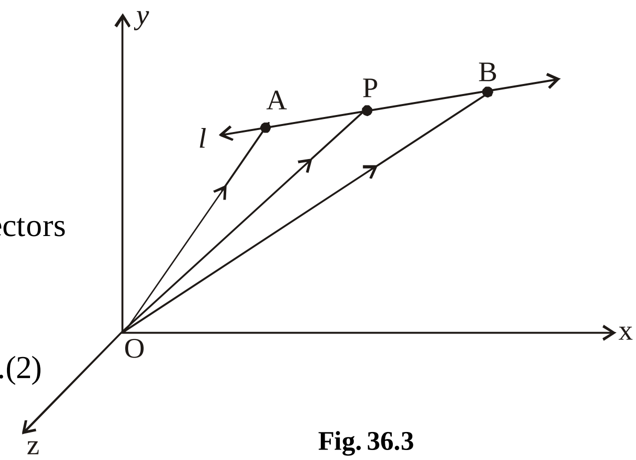


Fig. 36.3

which is the required equation in vector form.

36.1.4 CONVERSION OF VECTOR FORM INTO CARTESIAN FORM

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of point A and B respectively. Consider (x, y, z) as the coordinates of point P.

Then $$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

and $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Substituting these values in equation (2) we get

$$(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = \lambda(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

$$\Rightarrow \quad \frac{x - x_1}{x_2 - x_1} = \lambda, \frac{y - y_1}{y_2 - y_1} = \lambda, \frac{z - z_1}{z_2 - z_1} = \lambda$$

$\therefore \quad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$ is the corresponding Cartesian form of equation of the line. This is known as two point form of equation of line.

Example 36.1 Find the vector equation of the line through the point (2, –3, 5) and parallel to the vector $\hat{i} + 2\hat{j} - 3\hat{k}$.

Solution : Here $$\vec{a} = 2\hat{i} - 3\hat{j} + 5\hat{k}$$

and $\qquad \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$

$\therefore \qquad \vec{r} = (2\hat{i} - 3\hat{j} + 5\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$

which is the required equation of the line.

Example 36.2 Find the vector equation of a line passing through the points (–1, 5, 2) and (4, 3, –5).

Solution : Vector equation of line in two point form is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

Here $\qquad \vec{a} = -\hat{i} + 5\hat{j} + 2\hat{k}$

and $\qquad \vec{b} = 4\hat{i} + 3\hat{j} - 5\hat{k}$

$\therefore \qquad \vec{b} - \vec{a} = 5\hat{i} - 2\hat{j} - 7\hat{k}$

Hence, the required equation is $\qquad \vec{r} = (-\hat{i} + 5\hat{j} + 2\hat{k}) + \lambda(5\hat{i} - 2\hat{j} - 7\hat{k})$

Example 36.3 Write the following equation of a line in vector form $\dfrac{x+3}{2} = \dfrac{y-2}{-3} = \dfrac{z-5}{7}$.

Solution : Comparing the given equation with $\dfrac{x - x_1}{b_1} = \dfrac{y - y_1}{b_2} = \dfrac{z - z_1}{b_3}$

We get $\qquad x_1 = -3,\ y_1 = 2,\ z_1 = 5$
$\qquad b_1 = 2,\ b_2 = -3,\ b_3 = 7$

$\therefore \qquad \vec{a} = (-3\hat{i} + 2\hat{j} + 5\hat{k})$

and $\qquad \vec{b} = (2\hat{i} - 3\hat{j} + 7\hat{k})$

Hence, $\qquad \vec{r} = (-3\hat{i} + 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} - 3\hat{j} + 7\hat{k})$ is the required equation in vector form.

Example 36.4 Find the equations of the line through the point $(1, 2, -3)$ with direction cosines

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

Solution : The equations of the line are

$$\frac{x-1}{\frac{1}{\sqrt{3}}} = \frac{y-2}{\frac{1}{\sqrt{3}}} = \frac{z+3}{-\frac{1}{\sqrt{3}}}$$

or $$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z+3}{-1}$$

or $$x-1=y-2=-(z+3)$$

Example 36.5 Find the equations of a line passing through the point $(1,-3,2)$ and having direction ratios $(1,-2,3)$

Solution : The equations of the line are

$$\frac{x-1}{1}=\frac{y+3}{-2}=\frac{z-2}{3}$$

Example 36.6 Find the equations of the line passing through two points $(1,-3,2)$ and $(4,2,-3)$

Solution : The equations of the required line are

$$\frac{x-1}{4-1}=\frac{y+3}{2+3}=\frac{z-2}{-3-2} \quad \text{or} \quad \frac{x-1}{3}=\frac{y+3}{5}=\frac{z-2}{-5}$$

Example 36.7 Find the equations of the line passing through the points $(1,-5,-6)$ and parallel to the line joining the points $(0,2,3)$ and $(-1,3,7)$.

Solution : Direction ratios of the line joining the points $(0,2,3)$ and $(-1,3,7)$ are

$$-1-0,\ 3-2,\ 7-3$$

or $$-1,+1,+4$$

$\therefore$ Direction ratios of a line parallel to this line can be taken as $-1,1,4$.

Thus, equations of the line through the point $(1,-5,-6)$ and parallel to the given line are

$$\frac{x-1}{-1}=\frac{y+5}{1}=\frac{z+6}{4}$$

CHECK YOUR PROGRESS 36.1

1. Find the equations, in symmetric form, of the line passing through the point $(1,-2,3)$ with direction ratios $3,-4,5$.
2. Find the equations of the line, in symmetric form, passing through the points $(3,-9,4)$ and $(-9,5,-4)$.
3. Find the equations of the line, in symmetric form, passing through the points $(-7,5,3)$ and $(2,6,8)$

4. Find the equations of the line, in symmetric form, through the point (1,2,3) and parallel to the line joining the points $(-4, 7, 2)$ and $(5, -3, -2)$
5. Find the equations of a line passing through the origin and equally inclined to the co-ordinate axes.
6. Write the vector equation of the line passing through origin and (5, –2, 3).
7. Write the following equation of a line in vector form $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-3}{2}$.
8. Write the following equation of a line in Cartesian form :
$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$
9. Find the vector equation of a line passing through the point (2, –1, 4) and parallel to the vector $\hat{i}+2\hat{j}-\hat{k}$.

36.2 REDUCTION OF THE EQUATIONS OF A LINE INTO SYMMETRIC FORM

You may recall that a line can be thought of as the intersection of two non-parallel planes.

Let the equations of the two intersecting planes be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

and $$a'x + b'y + c'z + d' = 0 \quad \text{....(ii)}$$

Let AB be the line of intersection of the two planes. Every point on the line AB lies on both the planes. Thus, the co-ordinates of any point on the line satisfy the two equations of the planes. Hence (i) and (ii) together represent the equations of a line.

The equations $ax + by + cz = 0$ and $a'x + b'y + c'z = 0$ together represent the equations of the line through the origin parallel to the above line as the above two planes also pass through origin. The above form of the equations of a line is referred to as general (***or non-symmetric***) form of the equations of a line.

To reduce the general equations of a line given by (i) and (ii) in the symmetric form, we need the direction cosines of the line as well as the co-ordinates of a point on the line.

Let the direction cosines of the line be l, m and n. The line is perpendicular to the normal to planes given by (i) and (ii).

$\therefore \quad al + bm + cn = 0 \quad$ and $\quad a'l + b'm + c'n = 0$

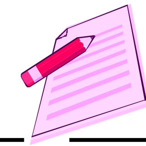
By cross multiplication method, we get

$$\frac{l}{bc'-b'c}=\frac{m}{ca'-ac'}=\frac{n}{ab'-a'b}$$

Thus, the direction cosines of the line are proportional to

$$(bc'-b'c), (ca'-ac') \text{ and } (ab'-a'b).$$

The point where the line meets the XY – plane is obtained by putting z = 0 in the equations (i) and (ii), which give



$$ax + by + d = 0 \qquad \text{....(iii)}$$

$$a'x + b'y + d' = 0 \qquad \text{....(iv)}$$

Solving (iii) and (iv), we get

$$x = \frac{bd' - b'd}{ab' - a'b}, \quad y = \frac{da' - d'a}{ab' - a'b}$$

$\therefore$ A point on the line is $\left(\dfrac{bd' - b'd}{ab' - a'b}, \dfrac{da' - d'a}{ab' - a'b}, 0\right)$

$\therefore$ The equations of the line in symmetric form are

$$\frac{x - \dfrac{bd' - b'd}{ab' - a'b}}{bc' - b'c} = \frac{y - \dfrac{da' - d'a}{ab' - a'b}}{ca' - c'a} = \frac{z}{ab' - a'b}$$

Note : Instead of taking z = 0, we may take x = 0 or y = o or any other suitable value for any of the x, y, z provided the two equations so obtained have a unique solution.

Example 36.8 Convert the equations of the line given by $x - 2y + 3z = 4,\ 2x - 3y + 4z = 5$ into symmetric form and find its direction cosines.

Solution : Let z = 0 be the z – co-ordinate of a point on each of the planes.

$\therefore$ The equations of the planes reduce to

$$x - 2y = 4$$

$$2x - 3y = 5$$

which on solving give $x = -2$ and $y = -3$

$\therefore$ The point common to two planes is $(-2, -3, 0)$.

Let *l*, m, n be the direction cosines of the line As the line is perpendicular to normal to the planes. we have

$$l - 2m + 3n = 0$$

and $$2l - 3m + 4n = 0$$

$$\therefore \quad \frac{l}{-8 + 9} = \frac{m}{6 - 4} = \frac{n}{-3 + 4}$$

or $$\frac{l}{1} = \frac{m}{2} = \frac{n}{1} = \pm\frac{1}{\sqrt{6}}$$

$\therefore$ The equations of the line are

$$\frac{x + 2}{\pm\dfrac{1}{\sqrt{6}}} = \frac{y + 3}{\pm\dfrac{2}{\sqrt{6}}} = \frac{z}{\pm\dfrac{1}{\sqrt{6}}}$$

or
$$\frac{x+2}{1}=\frac{y+3}{2}=\frac{z}{1}$$

and the direction cosines of the line are $\pm\frac{1}{\sqrt{6}}, \pm\frac{2}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}$ (the same sign positive or negative to be taken throughout)

CHECK YOUR PROGRESS 36.2

1. Find the equations, in symmetric form, of the line given by

(i) $x+5y-z=7$	and	$2x-5y+3z=-1$
(ii) $x+y+z+1=0$	and	$4x+y-2z+2=0$
(iii) $x-y+z+5=0$	and	$x-2y-z+2=0$

36.3 PERPENDICULAR DISTANCE OF A POINT FROM A LINE

Let P be the point (x_1, y_1, z_1) and AQ be the given line whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

where l, m and n are the direction cosines of the line AQ, Q is the foot of the perpendicular from P on AQ and A is the point (α, β, γ).

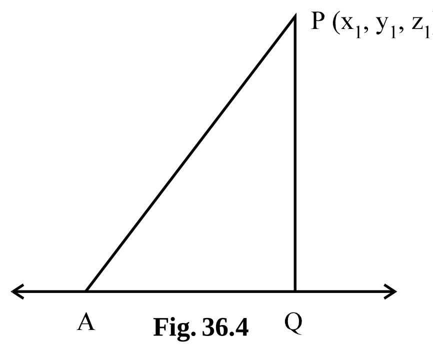


Fig. 36.4

We have $\quad PQ^2 = AP^2 - AQ^2$

Now $\quad AP^2 = (x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2$

Again AQ, the projection of AP on the line is

$(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n$

$$\therefore \quad PQ^2 = \left\{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2\right\}$$
$$-\{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n\}^2$$

which gives the length of perpendicular (PQ) from the point P to the line.

Example 36.9 Find the distance of a point (2,3,1) from the line

$$y+z-1=0=2x-3y-2z+4$$

Solution : Let z = 0 be the z-coordinate of the point common to two planes.

$\therefore$ Their equations become $y=1$ and $2x-3y+4=0$ which give $x=-\frac{1}{2}$

$\therefore$ A point common to two planes is $\left(-\frac{1}{2}, 1, 0\right)$

Let l, m, n be the direction cosines of the given line

Then, $0l + m + n = 0$ and $2l - 3m - 2n = 0$

or $\qquad \frac{l}{1} = \frac{m}{2} = \frac{n}{-2} = \frac{1}{\pm 3}$ or $\qquad l = \pm\frac{1}{3},\ m = \pm\frac{2}{3},\ n = \mp\frac{2}{3}$

If PQ is the length of the perpendicular from (2,3,1) to the given line. Then

$$PQ^2 = \left[\left(2 + \frac{1}{2}\right)^2 + (3-1)^2 + (1-0)^2\right] - \left[\frac{5}{2} \times \frac{1}{3} + \frac{2}{3} \times 2 - 1 \times \frac{2}{3}\right]^2$$

$$= \left(\frac{25}{4} + 4 + 1\right) - \left(\frac{5}{6} + \frac{4}{3} - \frac{2}{3}\right)^2$$

$$= \frac{45}{4} - \frac{9}{4} = 9$$

$\therefore$ PQ = 3

Thus, the required distance is 3 units.

CHECK YOUR PROGRESS 36.3

1. Find the distance of the point from the line, for each of the following :

   (i) Point (0, 2, 3), line $\frac{x+3}{3} = \frac{y-1}{2} = \frac{z+4}{3}$

   (ii) Point $(-1, 3, 9)$, line $\frac{x-13}{5} = \frac{y+8}{-6} = \frac{z-31}{1}$

   (iii) Point (4, 1, 1), line $x + y + z = 4,\ x - 2y - z = 4$

   (iv) Point (3, 2, 1), line $x + y + z = 4,\ x - 2y - z = 4$

36.4 ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and normal to the plane. Let the equations of the line be

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n} \qquad .....(i)$$

and that of the plane be

$$ax + by + cz + d = 0 \qquad ....(ii)$$

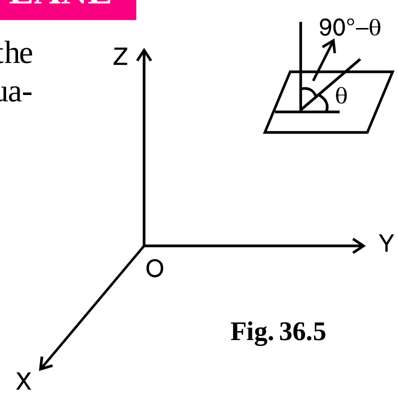


Fig. 36.5

If θ be the angle between (i) and (ii), then

$$\sin\theta = \cos(90^\circ - \theta) = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\ \sqrt{a^2 + b^2 + c^2}}$$

Example 36.10 Find the angle between the line $\frac{x-2}{3} = \frac{y+3}{3} = \frac{z-1}{1}$

and the plane $2x - 3y + 4z - 7 = 0$

Solution : Here the angle θ between the given line and given plane is given by

$$\sin\theta = \frac{2\times 3 - 3\times 3 + 4\times 1}{\sqrt{3^2 + 3^2 + 1^2}\ \sqrt{2^2 + (-3)^2 + 4^2}} = \frac{1}{\sqrt{19}\ \sqrt{29}}$$

$$= \frac{1}{\sqrt{551}}$$

or $$\theta = \sin^{-1}\left(\frac{1}{\sqrt{551}}\right)$$

CHECK YOUR PROGRESS 36.4

1. Find the angle between the following lines and the planes.

(i) Line : $\frac{x-4}{1} = \frac{y+2}{4} = \frac{z-3}{-1}$ and Plane : $3x - 4y + 5z = 5$

(ii) Line : $\frac{x-2}{2} = \frac{z-3}{3} = \frac{y+2}{1}$ and Plane : $-2x + 4y - 5z = 20$

(iii) Line : $\frac{x}{4} = \frac{y-2}{-3} = \frac{y+2}{5}$ and Plane : $x - 4y + 6z = 11$

(iv) Line : $\frac{x+2}{4} = \frac{y-3}{5} = \frac{z+4}{1}$ and Plane : $4x - 3y - z - 7 = 0$

36.5 CONDITION OF COPLANARITY OF TWO LINES

If the two lines given by

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \qquad \text{......(i)}$$

and $$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \qquad \text{.....(ii)}$$

intersect, they lie in the same plane.

Equation of a plane containing line (i) is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \qquad \text{......(iii)}$$

with $\quad Al_1 + Bm_1 + Cn_1 = 0$ .....(iv)

If the plane (iii) contains line (ii), the point (x_2, y_2, z_2) must lie on it.

Thus, $\quad A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0$ .....(v)

with $\quad Al_2 + Bm_2 + Cn_2 = 0$ ....(vi)

Eliminating A,B and C from (iv), (v) and (vi), we have

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad .....(vii)$$

which is the necessary condition for coplanarity of lines given by (i) and (ii)

Again, eliminating A,B and C from (iii), (iv) and (vi) we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad .....(viii)$$

(viii) represents the equation of the plane containing the two intersecting lines.

We shall now show that if the condition (vii) holds, then the lines (i) and (ii) are coplanar.

Consider the plane

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad ....(ix)$$

or, $\quad (x - x_1)(m_1n_2 - m_2n_1) + (y - y_1)(n_1l_2 - n_2l_1)$

$$+ (z - z_1)(l_1m_2 - l_2m_1) = 0$$

A line will lie in the plane, if the normal to the plane is perpendicular to the line and any point on the line lies in the plane.

You may see that

$$l_1(m_1n_2 - m_2n_1) + m_1(n_1l_2 - n_2l_1) + n_1(l_1m_2 - l_2m_1) = 0$$

Hence line (i) lies in plane (ix)

By similar argument, we can say that line (ii) lies on plane (ix)

$\therefore$ The two lines are coplanar.

Thus, the condition (vii) is also sufficient for the two lines to be coplanar.

Corollary : The lines (i) and (ii) will intersect if and only if (vii) holds and lines are not parallel.

Note :

(i) Two lines in space, which are neither intersecting nor parallel, do not lie in the same plane. Such lines are called skew lines.

(ii) If the equation of one line be in symmetric form and the other in general form, we proceed as follows:

Let equations of one line be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad \text{.... (i)}$$

and that of the other line be

$$ax + by + cz + d = 0 \text{ and } a'x + b'y + c'z + d' = 0 \qquad \text{......(ii)}$$

If the two lines are coplanar, then a point on the first line should satisfy equations of the second line. A general point on line (i) is $(x_1 + lr, y_1 + mr, z_1 + nr)$.

This point lies on $ax + by + cz + d = 0$ if

$$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0$$

or
$$r = -\frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn}$$

Similarly, this point should lie on $a'x + b'y + c'z + d' = 0$, resulting in

$$r = -\frac{a'x_1 + b'y_1 + c'z_1 + d'}{a'l + b'm + c'n}$$

Equating the two values of r obtained above, we have the required condition as

$$\frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn} = \frac{a'x_1 + b'y_1 + c'z_1 + d'}{a'l + b'm + c'n}$$

Note : In case, both the lines are in general form, convert one of them into symmetric form and then proceed as above.

Example 36.11 Prove that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ are co-planar.

Solution : For the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ ......(i)

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ .... (ii)

to be coplanar we must have

$$\begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

or $\quad 3(12+5)+3(12+35)+8(4-28)=0$

or $\quad 51+141-192=0$

or $\quad 0=0$ which is true.

$\therefore$ The two lines given by (i) and (ii) are coplanar.

Example 36.12 Prove that the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7} \text{ and } \frac{x-2}{1}=\frac{y-4}{4}=\frac{z-6}{7}$$

are coplanar. Find the equation of the plane containing these lines.

Solution : For the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7} \text{ and } \frac{x-2}{1}=\frac{y-4}{4}=\frac{z-6}{7}$$

to be coplanar, we must have

$$\begin{vmatrix} 2+1 & 4+3 & 6+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

or $\quad 3(35-28)-7(21-7)+11(12-5)=0$

or $\quad 21-98+77=0$

or $\quad 0=0.$ which is true.

$\therefore$ The given lines are coplanar.

Equation of the plane containing these lines is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

or $\quad (x+1)(35-28)-(y+3)(21-7)+(z+5)(12-5)=0$

or $\quad 7x+7-14y-42+7z+35=0$

or $\quad 7x-14y+7z=0$

or $\quad x-2y+z=0$

CHECK YOUR PROGRESS 36.5

1. Prove that the following lines are coplanar :

(i) $\frac{x-3}{3}=\frac{y-2}{-4}=\frac{z+1}{1}$ and $x+2y+3z=0=2x+4y+3z+3$

(ii) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $4x-3y+1=0=5x-3z+2$

2. Show that the lines $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$

and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ are coplanar. Find the equation of the plane containing them.

LET US SUM UP

A line is the intersection of two non-parallel planes.

- Vector equation of a line is $\vec{r}=\vec{a}+\lambda\vec{b}$, where $\vec{a}$ is the position vector of the given point on the line and $\vec{b}$ is a vector parallel to the line.
- Its corresponding Cartesian form is

  $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$, where (x_1, y_1, z_1) are the coordinates to the given point on the line and b_1, b_2, b_3 are the direction ratios of the vector $\vec{b}$.
- $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$ is another vector equation of the line where $\vec{a}$ and $\vec{b}$ are the position vectors of two distinct points on the line.
- Its corresponding Cartesian form is

  $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$, where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of two distinct given points on the line.

- The angle θ between the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane

  $ax+by+cz+d=0$ is given by

$$\sin\theta=\frac{al+bm+cn}{\sqrt{l^2+m^2+n^2}\sqrt{a^2+b^2+c^2}}$$

- The condition of coplanarity of two lines,

$$\frac{x-x_1}{l}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

and the equation of the plane containing the lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

SUPPORTIVE WEB SITES

http://www.regentsprep.org/regents/math/algebra/ac1/eqlines.htm

http://www.purplemath.com/modules/strtlneq.htm

http://www.mathsteacher.com.au/year10/ch03_linear_graphs/02_gradient/line.htm

TERMINAL EXERCISE

1. Find the equations of the line passing through the points (1,4,7) and $(3,-2,5)$

2. Find the equations of the line passing through the point $(-1,-2,-3)$ and perpendicular to the plane $3x-4y+5z-11=0$

3. Find the direction cosines of the line which is perpendicular to the lines whose direction ratios are $1,-1,2$ and $2,1,-1$.

4. Show that the line segment joining the points (1,2,3) and (4,5,7) is parallel to the line segment joining the points $(-4,3,-6)$ and $(2,9,2)$

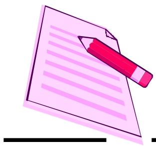
5. Find the angle between the lines

$$\frac{x-1}{2}=\frac{y-2}{-4}=\frac{z+5}{5} \quad \text{and} \quad \frac{x+1}{3}=\frac{y+1}{4}=\frac{z}{2}$$

6. Find the equations of the line passing through the point $(1,2,-4)$ and perpendicular to each of the two lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

7. Convert the equations of the line $x-y+2z-5=0$, $3x+y+z-6=0$ into the symmetric form.



8. Show that the lines $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z}{-1}$ and $\frac{x-4}{3}=\frac{y-1}{-2}=\frac{z-1}{1}$ are coplanar. Find the equation of the plane containing them.

9. Find the equation of the plane containing the lines.

$$\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5} \qquad \text{and} \qquad \frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$$

10. Find the projection of the line segment joining the points (2,3,1) and (5,8,7) on the line $\frac{x}{2}=\frac{y+4}{3}=\frac{z+1}{6}$

11. Find the vector equation of a line which passes through the point (1, 2, –4) and is parallel to the vector $(2\hat{i}+3\hat{j}-5\hat{k})$.

12. Cartesian equation of a line is $\frac{x+5}{3}=\frac{y-4}{-5}=z$, what is its vector equation?

13. Find the vector equation of a line passing through the points (3, –2, –5) and (3, –2, 6).

14. Find the vector equation of a line which passes through the point (–2, 4, –5) and parallel to the line given by $\frac{x-3}{3}=\frac{y+4}{5}=\frac{z-8}{2}$.

ANSWERS

CHECK YOUR PROGRESS 36.1

1. $\dfrac{x-1}{3}=\dfrac{y+2}{-4}=\dfrac{z-3}{5}$

2. $\dfrac{x-3}{-6}=\dfrac{y+9}{7}=\dfrac{z-4}{-4}$

3. $\dfrac{x+7}{9}=\dfrac{y-5}{1}=\dfrac{z-3}{5}$

4. $\dfrac{x-1}{9}=\dfrac{y-2}{-10}=\dfrac{z-3}{-4}$

5. $\dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z}{1}$

6. $\vec{r}=\lambda\left(5\hat{i}-2\hat{j}+3\hat{i}\right)$

7. $\vec{r}=\left(5\hat{i}-4\hat{j}+3\hat{i}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$

8. $\dfrac{x-1}{1}=\dfrac{y-2}{-3}=\dfrac{z-3}{2}$

9. $\vec{r}=\left(2\hat{i}-\hat{j}+4\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$

CHECK YOUR PROGRESS 36.2

1. (i) $\dfrac{x-2}{2}=\dfrac{y-1}{-1}=\dfrac{z}{-3}$

(ii) $\dfrac{x+\frac{1}{3}}{1}=\dfrac{y+\frac{2}{3}}{-2}=\dfrac{z}{1}$

(iii) $\dfrac{x-1}{3}=\dfrac{y-3}{2}=\dfrac{z+3}{1}$

CHECK YOUR PROGRESS 36.3

1. (i) $\sqrt{21}$ units (ii) 21 units

(iii) $\sqrt{\dfrac{27}{14}}$ units (iv) $\sqrt{6}$ units

CHECK YOUR PROGRESS 36.4

1. (i) $\sin^{-1}\left(-\dfrac{3}{5}\right)$ (ii) $\sin^{-1}\left(\dfrac{1}{\sqrt{70}}\right)$

(iii) $\sin^{-1}\left(\dfrac{46}{\sqrt{2650}}\right)$ (iv) 0°.

CHECK YOUR PROGRESS 36.5

2. $x + y + z = 0$

TERMINAL EXERCISE

1. $\dfrac{x-1}{2} = \dfrac{y-4}{-6} = \dfrac{z-7}{-2}$

2. $\dfrac{x+1}{3} = \dfrac{y+2}{-4} = \dfrac{z+3}{5}$

3. $-\dfrac{1}{\sqrt{35}}, \dfrac{5}{\sqrt{35}}, \dfrac{3}{\sqrt{35}}$

5. $90°$

6. $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z+4}{6}$

7. $\dfrac{x-\frac{11}{4}}{-3} = \dfrac{y+\frac{9}{4}}{5} = \dfrac{z}{4}$

8. $2x - 5y - 16z + 13 = 0$

9. $17x - 47y - 24z + 172 = 0$

10. $\dfrac{57}{7}$ units.

11. $\vec{r} = (\hat{i} + 2\hat{y} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 5\hat{k})$

12. $\vec{r} = (-5\hat{i} + 4\hat{j}) + \lambda(3\hat{i} - 5\hat{j} + \hat{k})$

13. $\vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + \lambda(11\hat{k})$

14. $\vec{r} = (-2\hat{i} + 4\hat{j} - 5\hat{k}) + \lambda(3\hat{i} + 5\hat{j} + 2\hat{k})$

37

LINEAR PROGRAMMING

37.1 INTRODUTION TO LINEAR PROGRAMMING PROBLEMS

A toy dealer goes to the wholesale market with Rs. 1500.00 to purchase toys for selling. In the market there are various types of toys available. From quality point of view, he finds that the toy of type 'A' and type 'B' are suitable. The cost price of type 'A' toy is Rs. 300 each and that of type 'B' is Rs. 250 each. He knows that the type 'A' toy can be sold for Rs. 325 each, while the type 'B' toy can be sold for Rs. 265 each. Within the amount available to him he would like to make maximum profit. His problem is to find out how many type 'A' and type 'B' toys should be purchased so to get the maximum profit.

He can prepare the following table taking into account all possible combinations of type 'A' and type 'B' toys subject to the limitation on the investment.

'A' type	'B' type	Investment	Amount after sale (including the unutilised amount if any)	Profit on the investment
0	6	1500.00	1590.00	90.00
1	4	1300.00	1585.00	85.00
2	3	1350.00	1595.00	95.00
3	2	1400.00	1605.00	105.00
4	1	1450.00	1615.00	115.00
5	0	1500.00	1625.00	125.00

Now, the decision leading to maximum profit is clear. Five type A toys should be purchased.

The above problem was easy to handle because the choice was limited to two types, and the number of items to be purchased was small. Here, all possible combinations were thought of and the corresponding gain calculated. But one must make sure that he has taken all possibilities into account.

A situation faced by a retailer of radio sets similar to the one given above is described below.

A retailer of radio sets wishes to buy a number of transistor radio sets from the wholesaler. There are two types (type A and type B) of radio sets which he can buy. Type A costs Rs.360 each and type *B* costs Rs. 240 each. The retailer can invest up to Rs. 5760. By selling the radio sets, he can make a profit of Rs. 50 on each set of type A and of Rs. 40 on each set of type B. How many of each type should he buy to maximize his total profit?

Here we have to maximize the profit. Sometimes we come across a problem in which the costs are to be minimized. Consider the following problem :

Two tailors A and B earn Rs.150 and Rs.200 per day respectively. A can stitch 6 shirts and 4 pants per day, while B can stitch 4 shirts and 7 pants per day. How many days shall each work if they want to produce at least 60 shirts and 72 pants at a minimum labour cost?

In this problem we have to minimise the labour cost.

These types of problems of maximisation and minimisation are called optimisation problems.

The technique followed by mathematicians to solve such problems is called **'Linear Programming'.**

OBJECTIVES

After studying this lesson, you will be able to :

- undertstand the terminology used in linear programming;
- convert different type of problems into a linear programming problem;
- use graphical mehtod to find solution of the linear programming problems

EXPECTED BACKGROUND KNOWLEDGE

- good idea of converting a mathematical information into a in equality
- to be able to solve system of on equalities using graphical method.

37.2 DEFINITIONS OF VARIOUS TERMS INVOLVED IN LINEAR PROGRAMMING

A close examination of the examples cited in the introduction points out one basic property that all these problems have in common, i.e., in each example, we were concerned with maximising or minimising some quantity.

In first two examples, we wanted to maximise the return on the investment. In third example, we wanted to minimise the labour cost. In linear programming terminology the maximization or minimization of a quantity is referred to as the objective of the problem.

37.2.1 OBJECTIVE FUNCTION

In a linear programming problem. z, the linear function of the variables which is to be optimized is called **objective function.**

Here, a linear form means a mathematical expression of the type

$$a_1x_1 + a_2\,x_2 + ... + a_nx_n\,,$$

where $a_1, a_{2,\,.....,}\,a_n$ are **constants** and $x_1, x_{2,\,.....,}\,x_n$ are **variables.**

In linear programming problems, the products, services, projects etc. that are competing with each other for sharing the given limited resources are called the variables or decision variables.

37.2.2 CONSTRAINTS

The limitations on resources (like cash in hand, production capacity, man power, time, machines, etc.) which are to be allocated among various competing variables are in the form of linear equations or inequations (inequalities) and are called **constraints** or **restrictions.**

37.2.3 NON-NEGATIVE RESTRICTIONS

All decision variables must assume non-negative values, as negative values of physical quantities is an impossible situation.

37.3 FORMULATION OF A LINEAR PROGRAMMING PROBLEM

The formulation of a linear programming problem as a mathematical model involves the following key steps.

Step 1 : Identify the decision variables to be determined and express them in terms of algebraic symbols such as $x_{1,} x_{2,} x_{3,}$ ....

Step 2 : Identify all the limitations in the given problem and then express them as linear equations or inequalities in terms of above defined decision variables.

Step 3 : Identify the objective which is to be optimised (maximised or minimised) and express it as a linear function of the above defined decision variables.

Example 37.1 A retailer wishes to buy a number of transistor radio sets of types *A* and *B*. Type *A* cost Rs.360 each and type *B* cost Rs. 240 each. The retailer knows that he cannot sell more than 20 sets, so he does not want to buy more than 20 sets and he cannot afford to pay more than Rs.5760. His expectation is that he would get a profit of Rs.50 for each set of type *A* and Rs.40 for each set of type *B*. Form a mathematical model to find how many of each type should be purchased in order to make his total profit as large as possible?

Solution : Suppose the retailer purchases x_1 sets of type *A* and x_2 sets of type *B*. Since the number of sets of each type is non-negative, so we have

$$x_1 \geq 0, \qquad \cdots(1)$$

$$x_2 \geq 0, \qquad \cdots(2)$$

Also the cost of x_1 sets of type *A* and x_2 sets of type *B* is 360 x_1 + 240 x_2 and it should be equal to or less than Rs.5760, that is,

$$360\,x_1 + 240\,x_2 \leq 5760$$

or $$3x_1 + 2x_2 \leq 48 \qquad \cdots(3)$$

Further, the number of sets of both types should not exceed 20, so

$x_1 + x_2 \leq 20 \qquad \cdots(4)$

Since the total profit consists of profit derived from selling the x_1 type *A* sets and x_2 type *B* sets, therefore, the retailer earns a profit of Rs.50 x_1 on type *A* sets and Rs.40 x_2 on type *B* sets. So the total profit is given by :

$$z = 50x_1 + 40x_2 \qquad \cdots(5)$$

Hence, the mathematical formulation of the given linear programming problem is as follows :

Find x_1, x_2 which

Maximise z = $50x_1 + 40x_2$ **(Objective function) subject to the conditions**

$$\left.\begin{array}{l} 3x_1 + 2x_2 \leq 48 \\ x_1 + x_2 \leq 20 \\ x_1 \geq 0, x_2 \geq 0 \end{array}\right\} \textbf{Constraints}$$

Example 37.2 A soft drink company has two bottling plants, one located at *P* and the other at *Q*. Each plant produ ces three different soft drinks *A*, *B*, and *C*. The capacities of the two plants in terms of number of bottles per day, are as follows :

Products ↓ \ Plants →	P	Q
A	3000	1000
B	1000	1000
C	2000	6000

A market survey indicates that during the month of May, there will be a demand for 24000 bottles of *A*, 16000 bottles of *B* and 48000 bottles of *C*. The operating cost per day of running plants *P* and *Q* are respectively Rs.6000 and Rs.4000. How many days should the firm run each plant in the month of May so that the production cost is minimised while still meeting the market demand.

Solution : Suppose that the firm runs the plant *P* for x_1 days and plant Q for x_2 days in the month of May in order to meet the market demand.

The per day operating cost of plant *P* is Rs.6000. Therefore, for x_1 days the operating cost will be Rs.6000 x_1.

The per day operating cost of plant *Q* is Rs.4000. Therefore, for x_2 days the operating cost will be Rs.4000 x_2.

Thus the total operating cost of two plants is given by :

$$z = 6000x_1 + 4000x_2 \qquad \cdots (1)$$

Plant P produces 3000 bottles of soft drink A per day. Therefore, in x_1 days plant P will produce 3000 x_1 bottles of soft drink A.

Plant Q produces 1000 bottles of soft drink A per day.

Therefore, in x_2 days plant Q will produce 1000 x_2 bottles of soft drink A.

Total production of soft drink A in the supposed period is 3000 x_1 + 1000 x_2

But there will be a demand for 24000 bottles of this soft drink, so the total production of this soft drink must be greater than or equal to this demand.

$\therefore \quad 3000x_1 + 1000x_2 \geq 24000$

or $\quad 3x_1 + x_2 \geq 24 \qquad \cdots (2)$

Similarly, for the other two soft drinks, we have the constraints

$$1000x_1 + 1000x_2 \geq 16000$$

or $\quad x_1 + x_2 \geq 16 \qquad \cdots (3)$

and

$$2000x_1 + 6000x_2 \geq 48000$$

or $\quad x_1 + 3x_2 \geq 24 \qquad \cdots (4)$

x_1 and x_2 are non-negative being the number of days, so

$$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots (5)$$

Thus our problem is to find x_1 and x_2 which

Minimize $z = 6000x_1 + 4000x_2$ ***(objective function)***

subject to the conditions

$$\left.\begin{aligned} 3x_1 + x_2 &\geq 24 \\ x_1 + x_2 &\geq 16 \\ x_1 + 3x_2 &\geq 24 \end{aligned}\right\} \qquad \textbf{\textit{(constraints)}}$$

and $x_1 \geq 0,\ x_2 \geq 0$

Example 37.3 A firm manufactures two types of products *A* and *B* and sells them at a profit of Rs.2 on type *A* and Rs.3 on type *B*. Each product is processed on two machines *G* and *H*. Type A requires one minute of processing time on *G* and 2 minutes on H, type *B* requires one minute on *G* and one minute on *H*. The machine *G* is available for not more than 6 hours and 40 minutes while machine *H* is available for 10 hours during one working day. Formulate the problem as a linear programming problem so as to maximise profit.

Solution : Let x_1 be the number of products of type *A* and x_2 be the number of products of type *B*.

The given information in the problem can systematically be arranged in the form of following table :

Machine	Processing time of the products (in minute)		Available time (in minute)
	Type *A* (x_1 units)	Type *B* (x_2 units)	
G	1	1	400
H	2	1	600
Profit per unit	Rs.2	Rs.3	

Since the profit on type *A* is Rs.2 per product, so the profit on selling x_1 units of type *A* will be $2x_{1.}$ Similarly, the profit on selling x_2 units of type *B* will be $3x_{2.}$ Therefore, total profit on selling x_1 units of type *A* and x_2 units of type *B* is given by

$$z = 2x_1 + 3x_2 \qquad \text{(objective function)} \qquad \cdots(1)$$

Since machine G takes 1 minute time on type *A* and 1 minute time on type *B*, therefore, the total number of minutes required on machine *G* is given by

$$x_1 + x_2$$

But the machine *G* is not available for more than 6 hours and 40 minutes (i.e., 400 minutes). Therefore,

$$x_1 + x_2 \le 400 \qquad \cdots(2)$$

Similarly, the total number of minutes required on machine *H* is given by

$$2x_1 + x_2$$

Also, the machine *H* is available for 10 hours (i.e., 600 minutes). Therefore,

$$2x_1 + x_2 \le 600 \qquad \cdots(3)$$

Since, it is not possible to produce negative quantities, so

$$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots(4)$$

Thus, the problem is to find x_1 and x_2 which

Maximize $\quad z = 2x_1 + 3x_2 \qquad$ (objective function)

subject to the conditions

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 600$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Example 37.4 A furniture manufacturer makes two types of sofas – sofa of type *A* and sofa of type *B*. For simplicity, divide the production process into three distinct operations, say carpentary, finishing and upholstery. The amount of labour required for each operation varies. Manufacture of a sofa of type *A* requires 6 hours of carpentary, 1 hour of finishing and 2 hours of upholstery. Manufacture of a sofa of type *B* requires 3 hours of carpentary, 1 hour of finishing and 6 hours of upholstery. Owing to limited availability of skilled labour as well as of tools and equipment, the factory has available each day 96 man hours of carpentary, 18 man hours for finishing and 72 man hours for upholstery. The profit per sofa of type *A* is Rs.80 and the profit per sofa of type *B* is Rs. 70. How many sofas of type *A* and type *B* should be produced each day in order to maximise the profit? Formulate the problems as linear programming problem.

Solution : The different operations and the availability of man hours for each operation can be put in the following tabular form :

Operations	Sofa of type *A*	Sofa of type *B*	Available labour
Carpentary	6 hours	3 hours	96 man hours
Finishing	1 hour	1 hour	18 man hours
Upholstery	2 hours	6 hours	72 man hours
Profit	Rs. 80	Rs. 70	

Let x_1 be the number of sofas of type *A* and x_2 be the number of sofas of type *B*.

Each row of the chart gives one restriction. The first row says that the amount of carpentary required is 6 hours for each sofa of type *A* and 3 hours for each sofa of type *B*. Further, only 96 man hours of carpentary are available per day. We can compute the total number of man hours of carpentary required per day to produce x_1 sofas of type *A* and x_2 sofas of type *B* as follows:

Number of man - hours per day of carpentary

= { (Number of hours carpentary per sofa of type *A*) $\times$ (Number of sofas of type A) }

+ { (Number of hours carpentary per sofa of type B) $\times$

(Number of sofas of type B) }

$$= 6x_1 + 3x_2$$

The requirement that at most 96 man hours of carpentary per day means

$$6x_1 + 3x_2 \leq 96$$

or $\quad \mathbf{2x_1 + x_2 \leq 32} \qquad \cdots$ **(1)**

Similarly, second and third row of the chart give the restrictions on finishing and upholstery respectively as

$\mathbf{x_1 + x_2 \leq 18} \qquad \cdots$ **(2)**

and

$$2x_1 + 6x_2 \leq 72$$

or $\quad \mathbf{x_1 + 3x_2 \leq 36} \qquad \cdots$ **(3)**

Since, the number of the sofas cannot be negative, therefore

$\mathbf{x_1 \geq 0, x_2 \geq 0} \qquad \cdots$ **(4)**

Now, the profit comes from two sources, that is, sofas of type A and sofas of type B. Therefore,

Profit = (Profit from sofas of type A) + (Profit from sofas of type B)

= { (Profit per sofa of type A) $\times$ (Number of sofas of type A) }

+ { (Profit per sofa of type B) $\times$ (Number of sofas of type B) }

$z = 80x_1 + 70x_2$ **(objective function)** $\cdots$ **(5)**

Thus, the problem is to find x_1 and x_2 which

Maximize $z = 80\, x_1 + 70\, x_2$ (objective function)

subject to the constraints

$$\left.\begin{array}{l} 2x_1 + x_2 \leq 32 \\ x_1 + x_2 \leq 18 \\ x_1 + 3x_2 \leq 36 \\ x_1 \geq 0,\ x_2 \geq 0 \end{array}\right\} \qquad \textbf{(Constraints)}$$

CHECK YOUR PROGRESS 37.1

1. A company is producing two products A and B. Each product is processed on two machines G and H. Type A requires 3 hours of processing time on G and 4 hours on H; type B requires 4 hours of processing time time on G and 5 hours on H. The available time is 18 hours and 21 hours for operations on G and H respectively. The products A and B can be sold at the profit of Rs. 3 and Rs. 8 per unit respectively. Formulate the problem as a linear programming problem.

2. A furniture dealer deals in only two items, tables and chairs. He has Rs. 5000 to invest and a space to store at most 60 pieces. A table costs him Rs. 250 and a chair Rs. 50. He can sell a table at a profit of Rs. 50 and a chair at a profit of Rs. 15. Assuming, he can sell all the items that he buys, how should he invest his money in order that may maximize his profit? Formulate a linear programming problem.

3. A dairy has its two plants one located at P and the other at Q. Each plant produces two types of products A and B in 1 kg packets. The capacity of two plants in number of packets per day are as follows:

Plants → Products ↓	P	Q
A	2000	1500
B	4000	6000

   A market survey indicates that during the month of April, there will be a demand for 20000 packets of A and 16000 packets of B. The operating cost per day of running plants P and Q are respectively Rs.4000 and Rs.7500. How many days should the firm run each plant in the month of April so that the production cost is minimized while still meeting the market demand? Formulate a Linear programming problem.

4. A factory manufactures two articles A and B. To manufacture the article A, a certain machine has to be worked for 1 hour and 30 minutes and in addition a craftsman has to work for 2 hours. To manufacture the article B, the machine has to be worked for 2 hours and 30 minutes and in addition the craftsman has to work for 1 hour and 30 minutes. In a week the factory can avail of 80 hours of machine time and 70 hours of craftsman's time. The profit on each article A is Rs.5 and that on each article B is Rs.4. If all the articles produced can be sold away, find how many of each kind should be produced to earn the maximum profit per week. Formulate the problem as a linear programming problem.

37.4 GEOMETRIC APPORACH OF LINEAR PROGRAMMING PROBLEM

Let us consider a simple problem in two variables x and y. Find x and y which satisfy the following equations

$$x + y = 4$$
$$3x + 4y = 14$$

Solving these equations, we get $x = 2$ and $y = 2$. What happens when the number of equations and variables are more?

Can we find a unique solution for such system of equations?

However, a unique solution for a set of simultaneous equations in n-variables can be obtained if there are exactly n-relations. What will happen when the number of relations is greater than or less then n ?

A unique solution will not exist, but a number of trial solutions can be found. Again, if the number of relations are greater than or less than the number of variables involved and the relation are in the form of inequalities.

Can we find a solution for such a system?

Whenever the analysis of a problem leads to minimising or maximising a linear expression in which the variable must obey a collection of linear inequalities, a solution may be obtained using linear programming techniques. One way to solve linear programming problems that involve only two variables is geometric approach called **graphical solution of the linear programming problem.**

37.5 SOLUTION OF LINEAR PROGRAMMING PROBLEMS

In the previous section we have seen the problems in which the number of relations are not equal to the number of variables and many of the relations are in the form of inequation (i.e., $\leq$ or $\geq$) to maximise (or minimise) a linear function of the variables subject to such conditions.

Now the question is how one can find a solution for such problems?

To answer this questions, let us consider the system of equations and inequations (or inequalities).

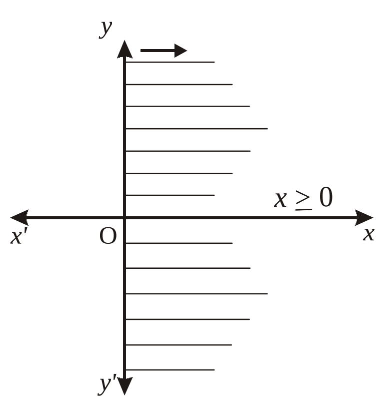


Fig. 37.1

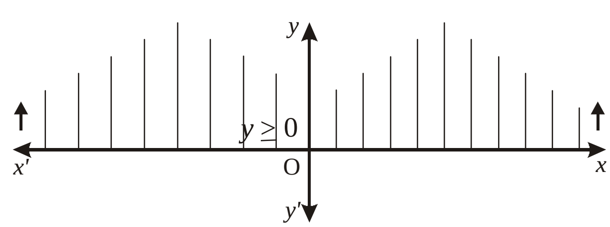


Fig. 37.2

We know that $x \geq 0$ represents a region lying towards the right of y - axis including the y - axis.

Similarly, the region represented by $y \geq 0$, lies above the x - axis including the x-axis.

The question arises: what region will be represented by $x \geq 0$ and $y \geq 0$ simultaneously.

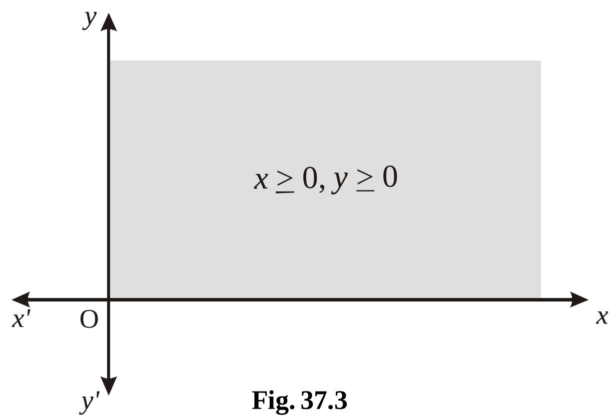


Fig. 37.3

Obviously, the region given by $x \geq 0$, $y \geq 0$ will consist of those points which are common to both $x \geq 0$ and $y \geq 0$. It is the first quadrant of the plane.

Next, we consider the graph of the equation $x + 2y \leq 8$. For this, first we draw the line $x + 2y = 8$ and then find the region satisfying $x + 2y \leq 8$.

Usually we choose x = 0 and calculate the corresponding value of y and choose y = 0 and calculate the corresponding value of x to obtain two sets of values (This method fails, if the line is parallel to either of the axes or passes through the origin. In that case, we choose any arbitrary value for x and choose y so as to satisfy the equation).

Plotting the points (0,4) and (8,0) and joining them by a straight line, we obtain the graph of the line as given in the Fig. 37.4 below.

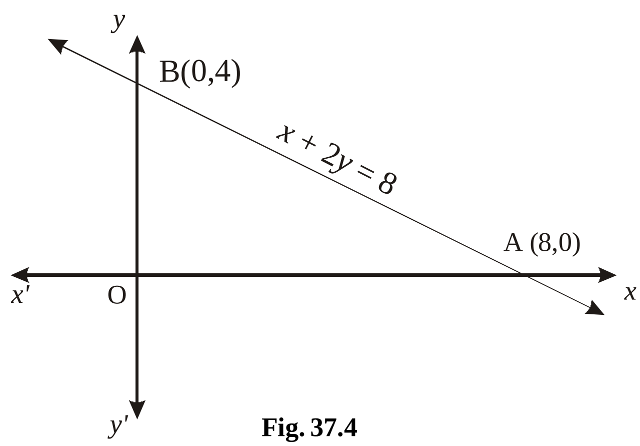


Fig. 37.4

We have already seen that $x \geq 0$ and $y \geq 0$ represents the first quadrant. The graph given by x + 2y < 8 lies towards that side of the line

$x + 2y = 8$ in which the origin is situated because any point in this region will satisfy the inequality. Hence the shaded region in the Fig. 37.5 represents $x \geq 0$, $y \geq 0$ and $x + 2y \leq 8$ simultaneously.

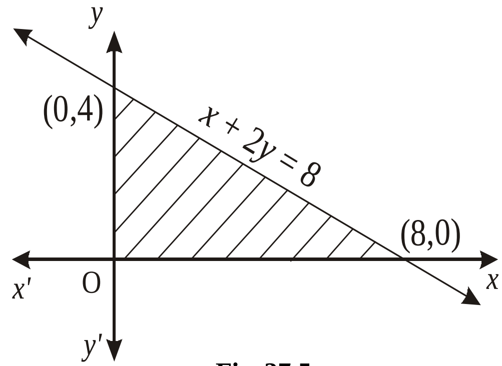


Fig. 37.5

Similarly, if we have to consider the regions bounded by $x \geq 0$, $y \geq 0$ and $x + 2y \geq 8$, then it will lie in the first quadrant and on that side of the line $x + 2y = 8$ in which the origin is not located. The graph is shown by the shaded region, in Fig. 37.6

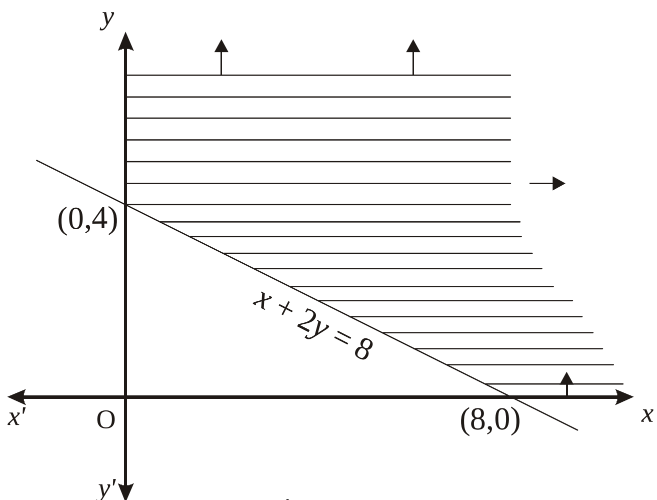


Fig. 37.6

The shaded region in which all the given constraints are satisfied is called the **feasible region.**

37.5.1 Feasible Solution

A set of values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called a **feasible solution** of the problem.

37.5.2 Optimal Solution

A feasible solution of a linear programming problem which optimises its objective functions is called the **optimal solution** of the problem.

Note : *If none of the feasible solutions maximise (or minimise) the objective function, or if there are no feasible solutions, then the linear programming problem has* ***no solution.***

In order to find a graphical solution of the linear programming problem, following steps be employed.

Step 1 : Formulate the linear programming problem.

Step 2 : Graph the constraints (inequalities), by the method discussed above.

Step 3 : Identify the feasible region which satisfies all the constraints simultaneously. For less than or equal to' constraints the region is generally below the lines and 'for greater than or equal to' constraints, the region is above the lines.

Step 4 : Locate the solution points on the feasible region. These points always occur at the vertex of the feasible region.

Step 5 : Evaluate the objective function at each of the vertex (corner point)

Step 6 : Identify the optimum value of the objective function.

Example 37.5 Minimise the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function to be minimised is

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities, which is as follows :

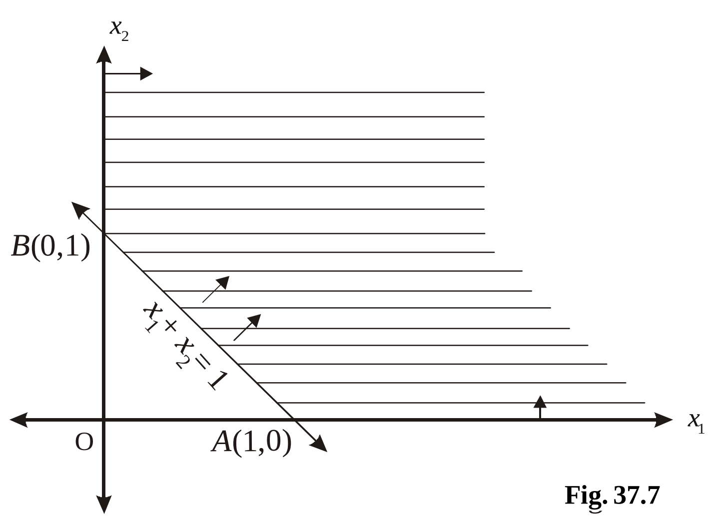


Fig. 37.7

As we have discussed earlier that the region satisfied by $x_1 \geq 0$ and $x_2 \geq 0$ is the first quadrant and the region satisfied by the line $x_1 + x_2 \geq 1$ along with $x_1 \geq 0,\ x_2 \geq 0$ will be on that side of the line $x_1 + x_2 = 1$ in which the origin is not located. Hence, the shaded region is our feasible solution because every point in this region satisfies all the constraints. Now, we have to find optimal solution. The vertex of the feasible region are A (1,0) and B (0,1).

The value of z at $A = 1$

The value of z at $B = 2$

Take any other point in the feasible region say (1,1), (2,0), (0,2) etc. We see that the value of z is minimum at A (1,0).

Example 37.6 Minimise the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function to be minimised is

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities (as discussed earlier) which is as follows :

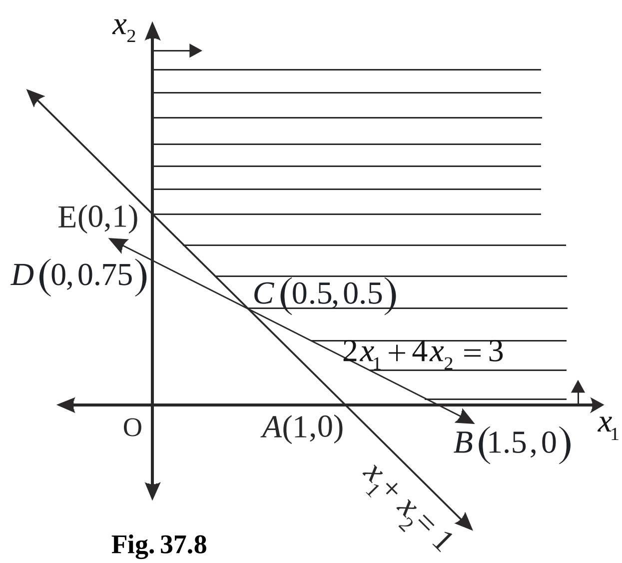


Fig. 37.8

The shaded region is the feasible region. Every point in the region satisfies all the mathematical inequalities and hence the feasible solution.

Now, we have to find the optimal solution.

The value of z at $B\ (1.5, 0)$ is 1.5

The value of z at $C\ (0.5, 0.5)$ is 1.5

The value of z at E (0,1) is 2

If we take any point on the line $2x_1 + 4x_2 = 3$ between B and C we will get $\frac{3}{2}$ and elsewhere in the feasible region greater than $\frac{3}{2}$. Of course, the reason any feasible point (between B and C) on $2x_1 + 4x_2 = 3$ minimizes the objective function (equation) $z = x_1 + 2x_2$ is that the two lines are parallel (both have slope $-\frac{1}{2}$). Thus this linear programming problem has infinitely many solutions and two of them occur at the vertices.

Example 37.7 Maximise

$$z = 0.25x_1 + 0.45x_2$$

subject to the constraints

$$x_1 + 2x_2 \le 300$$
$$3x_1 + 2x_2 \le 480$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Solution : The objective function is to maximise

$$z = 0.25x_1 + 0.45x_2$$

subject to the constraints

$$x_1 + 2x_2 \le 300$$
$$3x_1 + 2x_2 \le 480$$
$$x_1 \ge 0,\ x_2 \ge 0$$

First of all we draw the graphs of these inequalities, which is as follows :

The shaded region $OABC$ is the feasible region. Every point in the region satisfies all the mathematical inequations and hence the feasible solutions.

Now, we have to find the optimal solution.

The value of z at A (160, 0) is 40.00

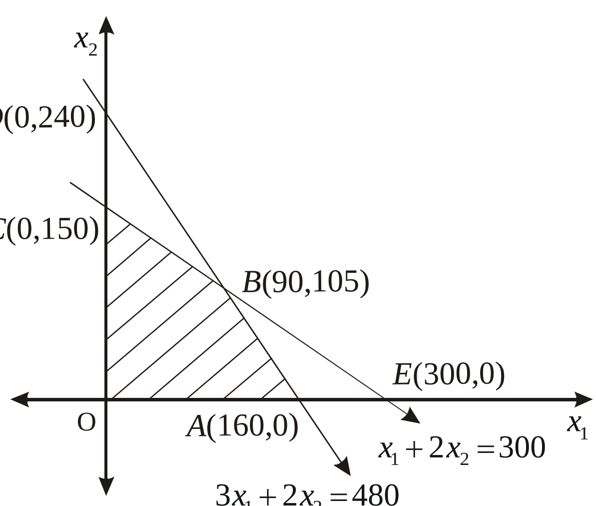


Fig. 37.9

The value of z at B (90, 105) is 69.75.

The value of z at C (0, 150) is 67.50

The value of z at O (0, 0) is 0.

If we take any other value from the feasible region say (60, 120), (80, 80) etc. we see that still the maximum value is 69.75 obtained at the vertex B (90, 105) of the feasible region.

Note : *For any linear programming problem that has a solution, the following general rule is true.*

If a linear programming problem has a solution it is located at a vertex of the feasible region. If a linear programming problem has multiple solutions, at least one of them is located at a vertex of the feasible region. In either case, the value of the objective function is unique.

Example 37.8 In a small scale industry a manufacturer produces two types of book cases. The first type of book case requires 3 hours on machine A and 2 hours on machines B for completion, whereas the second type of book case requires 3 hours on machine A and 3 hours on machine B. The machine A can run at the most for 18 hours while the machine B for at the most 14 hours per day. He earns a profit of Rs. 30 on each book case of the first type and Rs. 40 on each book case of the second type.

How many book cases of each type should he make each day so as to have a maximum porfit?

Solution : Let x_1 be the number of first type book cases and x_2 be the number of second type book cases that the manufacturer will produce each day.

Since x_1 and x_2 are the number of book cases so

$$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots \textbf{(1)}$$

Since the first type of book case requires 3 hours on machine A, therefore, x_1 book cases of first type will require $3x_1$ hours on machine A. second type of book case also requires 3 hours on machine A, therefore, x_2 book cases of second type will require $3x_2$ hours on machine A. But the working capacity of machine A is at most 18 hours per day, so we have

$$3x_1 + 3x_2 \leq 18$$

or $$x_1 + x_2 \leq 6 \qquad \cdots \textbf{(2)}$$

Similarly, on the machine B, first type of book case takes 2 hours and second type of book case takes 3 hours for completion and the machine has the working capacity of 14 hours per day, so we have

$$\mathbf{2x_1 + 3x_2 \le 14} \qquad \cdots \textbf{(3)}$$

Profit per day is given by

$$z = 30x_1 + 40x_2 \qquad \cdots \textbf{(4)}$$

Now, we have to determine x_1 and x_2 such that

Maximize $z = 30x_1 + 40x_2$ (objective function) subject to the conditions

$$\left.\begin{array}{l} x_1 + x_2 \le 6 \\ 2x_1 + 3x_2 \le 14 \\ x_1 \ge 0,\ x_2 \ge 0 \end{array}\right\} \qquad \textbf{constraints}$$

We use the graphical method to find the solution of the problem. First of all we draw the graphs of these inequalities, which is as follows :

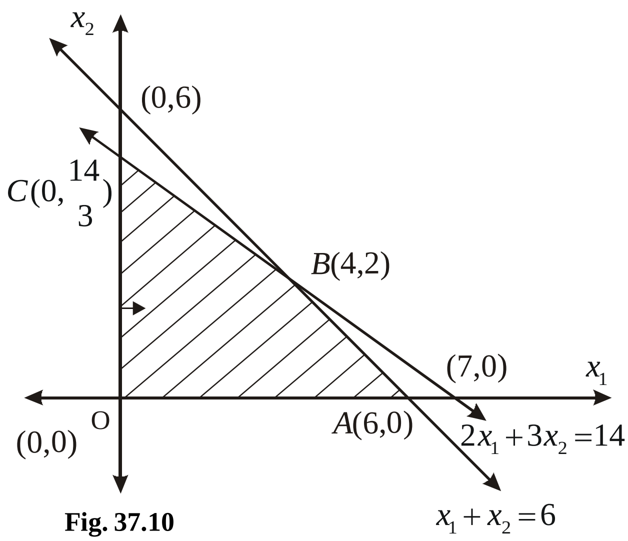


Fig. 37.10

The shaded region OABC is the feasible region. Every point in the region satisfies all the mathematical inequations and hence known as feasible solution.

We know that the optimal solution will be obtained at the vertices *O* (0, 0), *A* (6, 0). *B* (4,2). Since the co-ordinates of *C* are not integers so we don't consider this point. Co-ordinates of *B* are calculated as the intersection of the two lines.

Now the profit at *O* is zero.

Profit at A = 30 × 6 + 40 × 0

= 180

Profit at B = 30 × 4 + 40 × 2

$= 120 + 80$

$= 200$

Thus the small scale manufacturer gains the maximum profit of Rs.200 if he prepares 4 first type book cases and 2 second type book cases.

Example 37.9 Maximize the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

Solutions : First we graph the constraints

$$x_1 + x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

The shaded portion is the set of feasible solution.

Now, we have to maximize the objective function.

The value of z at $A(1, 0)$ is 1.

The value of z at $B(0, 1)$ is 2.

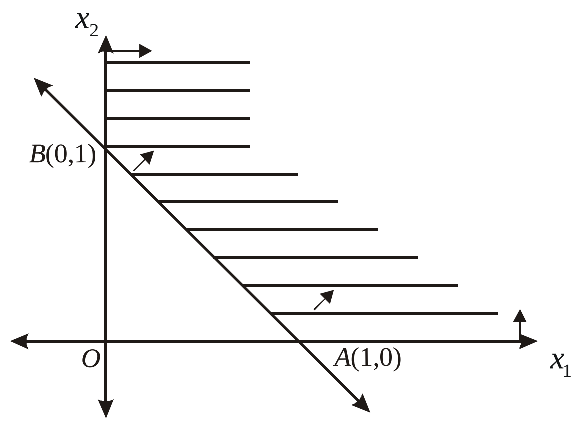


Fig. 37.11

If we take the value of z at any other point from the feasible region, say (1, 1) or (2, 3) or (5, 4) etc, then we notice that every time we can find another point which gives the larger value than the previous one. Hence, there is no feasible point that will make z largest. Since there is no feasible point that makes z largest, we conclude that this linear programming problem has no solution.

Example 37.10 Solve the following problem graphically.

Minimize $z = 2x_1 - 10x_2$

subject to the constraints

$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \leq -5$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : First we graph the constraints

$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \leq -5$$
$$x_1 \geq 0,\ x_2 \geq 0$$

$$x_2 - x_1 \le 0,$$

or $$5x_2 - x_1 \ge 5$$

$$x_1 - x_2 = 0$$

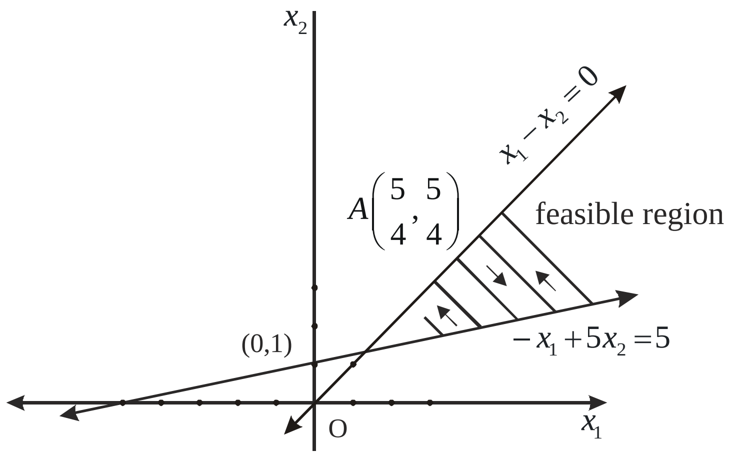


Fig. 37.12

The shaded region is the feasible region.

Here, we see that the feasible region is unbounded from one side.

But it is clear from Fig. 37.26 that the objective function attains its minimum value at the point A which is the point of intersection of the two lines $x_1 - x_2 = 0$ and $-x_1 + 5x_2 = 5$.

Solving these we get $x_1 = x_2 = \frac{5}{4}$

Hence, z is minimum when $x_1 = \frac{5}{4}$, $x_2 = \frac{5}{4}$, and its minimum value is

$$2 \times \frac{5}{4} - 10 \times \frac{5}{4} = -10.$$

***Note :** If we want to find max. z with these constraints then it is not possible in this case because the feasible region is unbounded from one side.*

CHECK YOUR PROGRESS 37.2

Solve the following problems graphically

1. **Maximize** $z = 3x_1 + 4x_2$

   subject to the conditions

2. **Maximize** $\neq = 2x_1 + 3x_2$

   subject to the conditions

$$x_1 + x_2 \le 40$$
$$x_1 + 2x_2 \le 60$$
$$x_1 \ge 0,\ x_2 \ge 0$$

$$x_1 + x_2 \le 400$$
$$2x_1 + x_2 \le 600$$
$$x_1 \ge 0, x_2 \ge 0$$

3. **Minimize** $z = 60x_1 + 40x_2$

subject to the conditions

$$3x_1 + x_2 \ge 24$$
$$x_1 + x_2 \ge 16$$
$$x_1 + 3x_2 \ge 24$$
$$x_1 \ge 0,\ x_2 \ge 0$$

4. **Maximize** $z = 20x_1 + 30x_2$

subject to the conditions

$$x_1 + x_2 \le 12,$$
$$5x_1 + 2x_2 \le 50$$
$$x_1 + 3x_2 \le 30,$$
$$x_1 \ge 0,\ x_2 \ge 0$$

5. **Maximize** $z = 50x_1 + 15x_2$

subject to the conditions

$$5x_1 + x_2 \le 100$$
$$x_1 + x_2 \le 60$$
$$x_1 \ge 0, x_2 \ge 0$$

6. **Minimize** $z = 4000x_1 + 7500x_2$

subject to the conditions

$$4x_1 + 3x_2 \ge 40$$
$$2x_1 + 3x_2 \ge 8$$
$$x_1 \ge 0, x_2 \ge 0$$

LET US SUM UP

- Linear programming is a technique followed by mathematicians to solve the optimisation problems.
- A set of values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called a **feasible solution**.
- A feasible solution of a linear programming problem which optimises its objective function is called the **Optimal solution** of the problem.
- The optimal solution of a linear programming problem is located at a vertex of the set of feasible region.
- If a linear programming problem has multiple solutions, at least one of them is located at a vertex of the set of feasible region. But in all the cases the value of the objective function remains the same.

SUPPORTIVE WEB SITES

http://people.brunel.ac.uk/~mastjjb/jeb/or/morelp.html
http://en.wikipedia.org/wiki/Simplex_algorithm
http://www.youtube.com/watch?v=XbGM4LjM52k

TERMINAL EXERCISE

1. A dealer has ₹ 1500 only for a purchase of rice and wheat. A bag of rice costs ₹ 1500 and a bag of wheat costs ₹ 1200. He has a storage capacity of ten bags only and the dealer gets a profit of ₹ 100 and ₹ 80 per bag of rice and wheat respectively. Formulate the problem as a linear programming problem to get the maximum profit.

2. A business man has ₹ 600000 at his disposal and wants to purchase cows and buffaloes to take up a business. The cost price of a cow is ₹ 20,000 and that of a buffalo is ₹ 60000. The man can store fodder for the live stock to the extent of 40 quintals per week. A cow gives 10 litres of milk and buffalo gives 20 litres of milk per day. Profit per litre of milk of cow is ₹ 5 and per litre of the milk of a buffalo is ₹ 7. If the consumption of fodder per cow is 1 quintal and per buffalo is 2 quintals a week, formulate the problem as a linear programming problem to find the number of live stock of each kind the man has to purchase so as to get maximum profit (assuming that he can sell all the quantity of milk, he gets from the livestock)

3. A factory manufactures two types of soaps each with the help of two machines A and B. A is operated for two minutes and B for 3 minutes to manufacture the first type, while the second type is manufactured by operating A for 3 minutes and B for 5 minutes. Each machine can be used for at most 8 hours on any day. The two types of soaps are sold at a profit of 25 paise and 50 paise each respectively. How many soaps of each type should the factory produce in a day so as to maximize the profit (assuming that the manufacturer can sell all the soaps he can manufacture). Formulate the problem as a linear programming problem.

4. Determine two non-negative rational numbers such that their sum is maximum provided that their difference exceeds four and three times the first number plus the second should be less than or equal to 9. Formulate the problem as a linear programming problem.

5. Vitamins A and B are found in two different foods E and F. One unit of food E contains 2 units of vitamin A and 3 units of vitamin B. One unit of food F contains 4 units of vitamin A and 2 units of vitamin B. One unit of food E and F costs Rs.5 and Rs.2.50 respectively. The minimum daily requirements for a person of vitamin A and B is 40 units and 50 units respectively. Assuming that anything in excess of daily minimum requirement of vitamin A and B is not harmful, find out the optimal mixture of food E and F at the minimum cost which meets the daily minimum requirement of vitamin A and B. Formulate this as a linear programming problem.

6. A machine producing either product A or B can produce A by using 2 units of chemicals and 1 unit of a compound and can produce B by using 1 unit of chemicals and 2 units of the compound. Only 800 units of chemicals and 1000 units of the compound are available. The profits available per unit of A and B are respectively Rs. 30 and Rs.20. Find the optimum allocation of units between A and B to maximise the total profit. Find the maximum profit.

7. Solve the following Linear programming problem graphically.

(a) Maximize $z = 25x_1 + 20x_2$

subject to the constraints

$$3x_1 + 6x_2 \le 50$$
$$x_1 + 2x_2 \le 10$$
$$x_1 \ge 0,\ x_2 \ge 0$$

(b) Maximize $z = 9x_1 + 10x_2$

subject to the constraints

$$11x_1 + 9x_2 \le 9900$$
$$7x_1 + 12x_2 \le 8400$$
$$3x_1 + 8x_2 \le 4800$$
$$x_1 \ge 0,\ x_2 \ge 0$$

(c) Maximise $z = 22x_1 + 18x_2$

subject to the constraints

$$x_1 + x_2 \le 20,$$
$$3x_1 + 2x_2 \le 48$$
$$x_1 \ge 0,\ x_2 \ge 0$$

ANSWERS

CHECK YOUR PROGRESS 37.1

1. Maximize $z = 3x_1 + 8x_2$

   subject to the constraints

   $3x_1 + 4x_2 \le 18$
   $4x_1 + 5x_2 \le 21$
   $x_1 \ge 0,\ x_2 \ge 0.$

2. Maximize $z = 50x_1 + 15x_2$

   subject to the constraints

   $5x_1 + x_2 \le 100$
   $x_1 + x_2 \le 60$
   $x_1 \ge 0,\ x_2 \ge 0.$

3. Minimize $z = 4000x_1 + 7500x_2$

   subject to the constraints

   $4x_1 + 3x_2 \ge 40$
   $2x_1 + 3x_2 \ge 8$
   $x_1 \ge 0,\ x_2 \ge 0$

4. Maximize $z = 5x_1 + 4x_2$

   subject to the constraints

   $1.5x_1 + 2.5x_2 \le 80$
   $2x_1 + 1.5\ x_2 \le 70$
   $x_1 \ge 0,\ x_2 \ge 0$

CHECK YOUR PROGRESS 37.2

1.

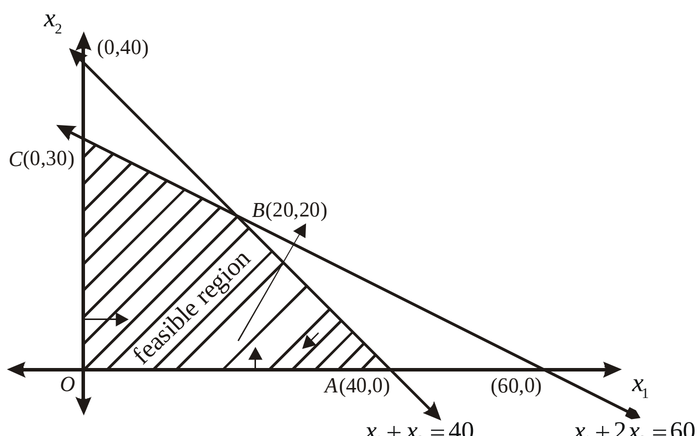


Fig. 37.13

Maximum $z = 140$ at $B(20,20)$

2.

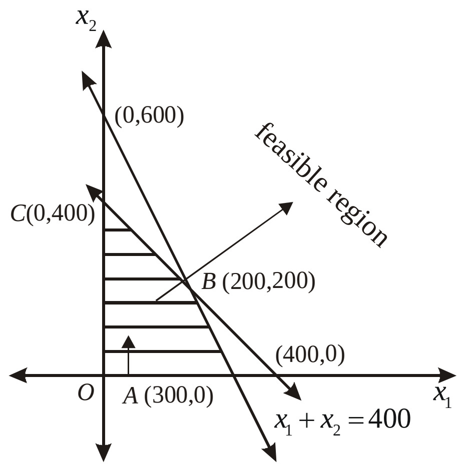


Fig. 37.14

Maximize $z = 1200$ at $C(0,400)$

3.

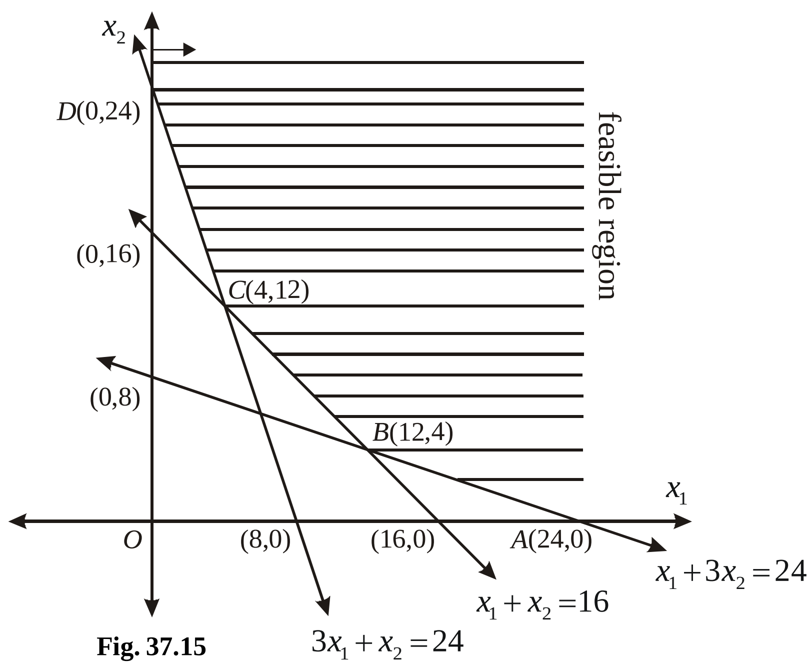


Fig. 37.15

Minimize $z = 720$ *at* $C(4,12)$, $x_1 = 4$, $x_2 = 12$

4.

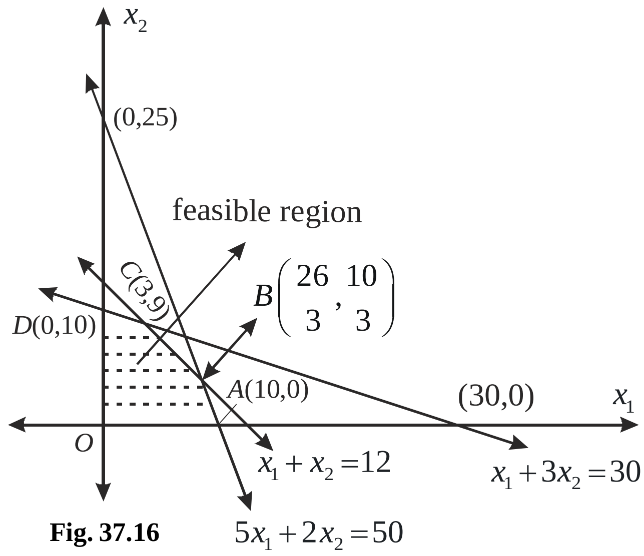


Fig. 37.16

Maximum $z = 330$ *at* $C(3,9)$, $x_1 = 3$, $x_2 = 9$

5.

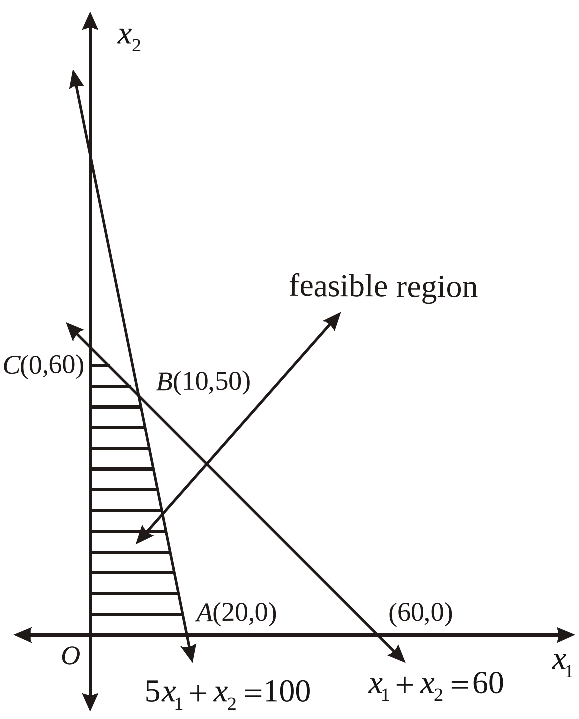


Fig. 37.17

Maximum $z = 1250$ *at* $B(10,50)$, $x_1 = 10$, $x_2 = 50$

MODULE - X
Linear Programming and Mathematical Reasoning

Notes

6.

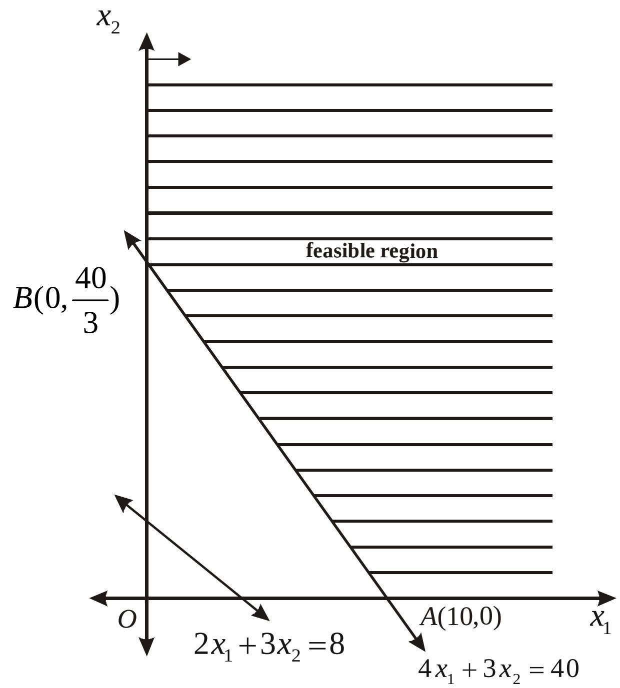


Fig. 37.18

Maximum z $= 40{,}000$ at A(10,0), $x_1 = 10,\ x_2 = 0$

TERMINAL EXERCISE

1. Maximize $z = 100x_1 + 80x_2$

   subject to the conditions

   $5x_1 + 4x_2 \leq 50$
   $x_1 + x_2 \leq 10$
   $x_1 \geq 0,\ x_2 \geq 0$

2. Maximize $z = 150x_1 + 980x_2$

   subject to the conditions

   $x_1 + 3x_2 \leq 30$
   $7x_1 + 14x_2 \leq 40$
   $x_1 \geq 0,\ x_2 \geq 0$

3. Maximize $z = 25x_1 + 50x_2$

   subject to the conditions

   $2x_1 + 3x_2 \leq 480$
   $3x_1 + 5x_2 \leq 480$
   $x_1 \geq 0,\ x_2 \geq 0$

4. Maximize $z = x_1 + x_2$

   subject to the conditions

   $x_1 - x_2 \geq 4$
   $3x_1 + x_2 \leq 9$
   $x_1 \geq 0,\ x_2 \geq 0$

5. Minimize $z = 5x_1 + 2.5x_2$

   subject to the conditions

$x_1 + 2x_2 \geq 20$

$3x_1 + 2x_2 \geq 50$

$x_1 \geq 0,\ x_2 \geq 0$

6. Maximize $z = 30x_1 + 20x_2$
subject to the constraints

$2x_1 + x_2 \leq 800$

$x_1 + 2x_2 \leq 1000$

$x_1 \geq 0,\ x_2 \geq 0$

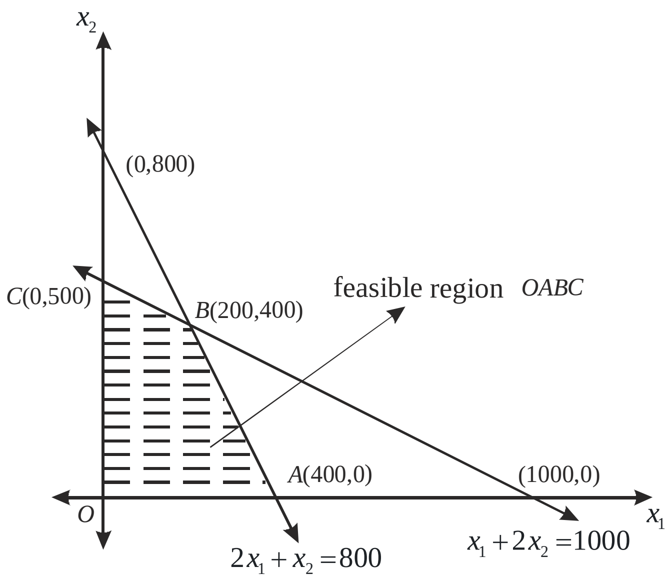


Fig. 37.19

Maximum z = 14000 at
B (200,400). $14000 = z$

8. (a)

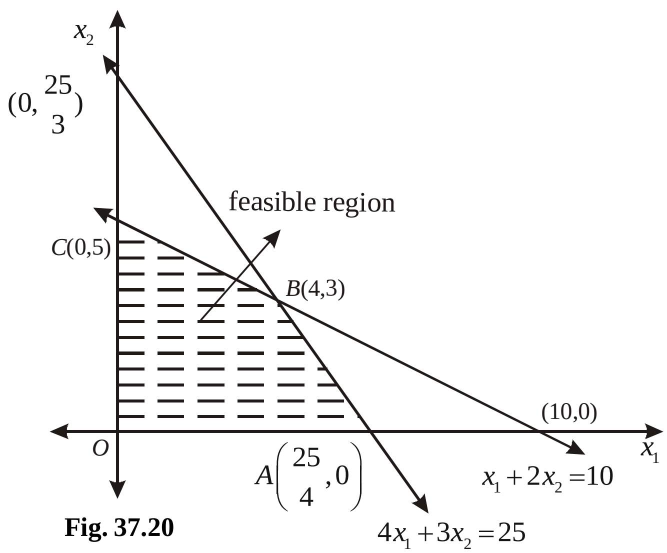


Fig. 37.20

Maximum $z = 160\ at\ B(4,3), x_1 = 4 x_2 = 3$

(b)

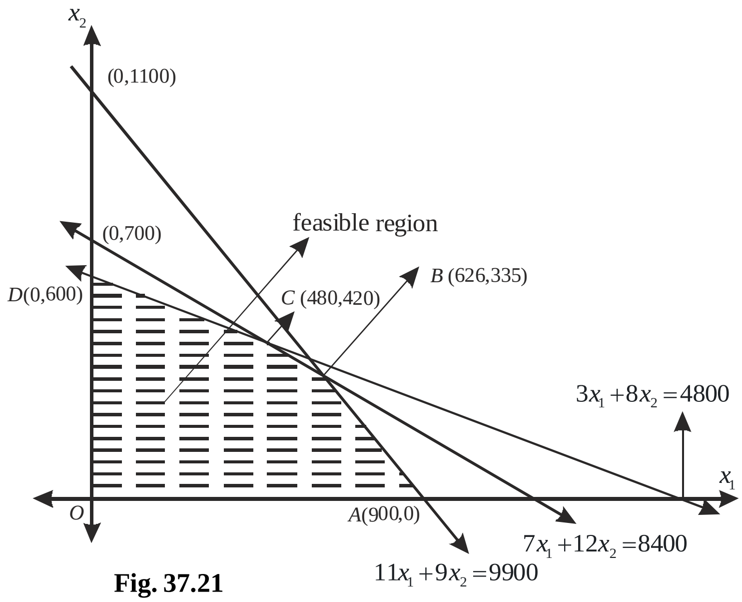


Fig. 37.21

A (900,0) D(0,600) B (626, 335), O(0, 0) and C (480, 420)

Maximum $z = 8984$ *at* $B(626, 335)$ $x_1 = 626,$ $x_2 = 335$

(c)

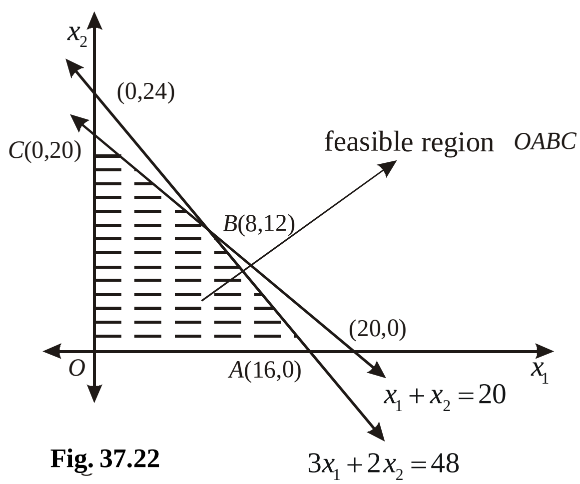


Fig. 37.22

Maximum $z = 392$ *at* $B(8,12)$ $x_1 = 8 x_2 = 12$

38

MATHEMATICAL REASONING

38.1 INTRODUCTION

In this lesson, we shall learn about some basic ideas of mathematical reasoning and the process of reasoning especially in context of mathematics. In mathematical language, there are two kinds of reasoning. (i) Inductive reasoning and (ii) Deductive reasoning. We have already discussed the inductive reasoning in mathematical induction. Now, we shall discuss some fundamentals of deductive reasoning.

38.2 STATEMENT (OR PROPOSITION)

The basic unit involved in mathematical reasoning is a mathematical statement :

> A sentence is called a mathematically acceptable statement if it is either true or false but not both at the same time.

If a statement is true, we say that it is a valid statement. A false statement is known as an invalid statement.

- Consider the following two sentences :

  Three plus four is 6.

  Two plus three is 5.

  When we read these sentences, we immediately decide that the first sentence is wrong and second is correct. There is no confusion regarding these. In mathematics such sentences are called statements.

- Now consider the following sentence :

  Mathematics is fun.

  Mathematics is fun is true for those who like mathematics. But, for others, it may not be true. So, the given sentence is true or false both. Hence, it is not a statement.

- Consider the following sentences :

  (i) Moon revolves around the Earth.

  (ii) Every square is a rectangle.

  (iii) The Sun is a Star.

  (iv) Every rectangle is a square.

  (v) New Delhi is in Pakistan

  When we read these sentences, the first, second and third sentences are true but fourth and fifth are-false sentences. Hence, each of them is a statement.

- Consider the following sentences :

(i) Give me a glass of water

(ii) Switch on the light

(iii) Where are you going?

(iv) How are you?

(v) How beautiful!

(vi) May you live long!

(vii) Tomorrow is Wednesday

We can not decide the truth value of (i), (ii), (iii), (iv), (v), (vi) and (vii). Hence, they are not statements.

Example 38.1 Check whether the following sentences are statements. Give reasons for your answer.

(i) 12 is less than 16.

(ii) Every set is a finite set.

(iii) $x + 5 = 11$.

(iv) There is no rain without clouds.

(v) All integers are natural numbers.

(vi) How far is Agra form here?

(vii) Are you going to Kanpur?

(viii) All roses are white.

Solution : (i) This sentences is true, because $12 < 16$ (12 is less than 16). Hence, it is a statement.

(ii) This sentence is false, because there are sets which are not finite. Hence, it is a statement.

(iii) The sentence $x + 5 = 11$ is an open sentence. Its truth value cannot be confirmed unless we are given the value of x. Hence, it is not a statement.

(iv) It is scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence, it is a statement.

(v) This sentence is false, because all integers are not natural numbers. So, it is a statement.

(vi) This sentence is a question (or interrogative sentence). Hence, it is not statement.

(vii) We can't have a truth value for it. So it is not a statement.

(viii) This sentence is false, because all roses are not white. Hence, it is a statement.

CHECK YOUR PROGRESS 38.1

1. Which of the following sentences are statements? Give reasons for your answer.

(i) Today is a windy day.

(ii) There are 40 days in a month.

(iii) The sum of 6 and 8 is greater than 12.

(iv) The square of a number is an even number

(v) Mathematics is difficult

(vi) All real numbers are complex numbers

(vii) The product of (–2) and (–5) is (–10).

(viii) There are 14 months in a year.

(ix) The real number x is less than 4

(x) Listen to me, Mohan!

(xi) Are all circles round?

(xii) All triangle have three sides.

38.3 NEGATION OF A STATEMENT

"The denial of a statement is called the negation of the statement."

Let us consider the statement :

P : New Delhi is a city.

The negation of this statement is

It is not the case that New Delhi is a city.

or

It is false that New Delhi is a city

or

New Delhi is not a city.

> If p is statement, then the negation of p is also a statement and is denoted by $\sim p$, and read as 'not p'.

Example 38.2 Write the negation of the following statements :

(i) Sum of 2 and 3 is 6.

(ii) $\sqrt{7}$ is rational.

(iii) Australia is a continent.

(iv) The number is less than 5.

Solution : (i) P : Sum of 2 and 3 is 6.

$\sim$P : Sum of 2 and 3 is not 6.

(ii) q : $\sqrt{7}$ is rational

$\sim q$: $\sqrt{7}$ is not rational

or

It is false that $\sqrt{7}$ is rational

(iii) r : Australia is a continent

~r : Australia is not a continent

(iv) S : The number 8 is less than 5.

~S : The number 8 is not less than 5.

or

It is false that the number 8 is less than 5.

38.4 COMPOUND STATEMENTS

In mathematical reasoning, we generally come across two types of statements.

(1) **Simple Statements :** A statement which cannot be broken into two or more statements is called a simple statement. For example :

(i) Every set is a finite set

(ii) New Delhi is the capital of India

(iii) Roses are white

(iv) $\sqrt{2}$ is an irrational number

(v) The set of real numbers is an infinite set.

(2) **Compound Statement :** A statement that can be formed by combining two or more simple statements is called a compound statement.

For example :

(i) Mohan is very smart or he is very lucky. This statement is actually made up of two statements connected by "or".

p : Mohan is very smart.

q : Mohan is very Lucky.

(ii) Sun is bigger than earth and earth is bigger than moon.

This statement is made up of two simple statements connected by 'and'.

p : Sun is bigger than earth.

q : Earth is bigger than moon.

Example 38.3 Find the component statements of the following compound statements.

(i) The sky is blue and the grass is green.

(ii) All rational number are real and all real numbers are complex.

(iii) It is raining and it is cold.

(iv) $\sqrt{2}$ is a rational number or an irrational number.

Solution : (i) The component statements are

p : The sky is blue

q : The grass is green

The connecting word is ‘and’.

(ii) The component statements are

p : All rational number are real

q : All real numbers are complex.

The connecting word is ‘and’.

(iii) The component statements are

p : It is raining

q : It is cold.

The connecting word is ‘and’

(iv) The component statements are

p : $\sqrt{2}$ is a rational number

q : $\sqrt{2}$ is an irrational number

The connecting word is ‘or’

Example 38.4 Find the component statements of the following compound statements.

(i) 0 is positive number or negative number.

(ii) All prime numbers are either even or odd.

(iii) Chandigarh is the capital of Panjab and U.P.

(iv) 12 is multiple of 2, 3 and 4.

Solution : (i) The component statements are

P : 0 is a positive number

q : 0 is a negative number

The connecting word is ‘or’.

(ii) The component statements are

p : All prime numbers are even numbers

q : All prime numbers are odd numbers

The connecting word is ‘or’

(iii) The component statements are

p : Chandigarh is the capital of Panjab.

q : Chandigarh is the capital of U.P.

The connecting word is ‘and’.

(iv) The component statements are

p : 12 is a multiple of 2

q : 12 is a multiple of 3

r : 12 is a multiple of 4

All the three statements are true. Here the connecting word is ‘and’.

38.5 IMPLICATIONS

In this section, we shall discuss the implication "if then", "only if", and "if and only if".

The statements with "if then" are very common in mathematics. For example, consider the statement.

r : If you are born in some country, then you are a citizen of that country.

We observed that if corresponds to two statements p and q given by

p : you are born in some country

q : you are citizen of that country

p and q are two statements forming the implication "if p then q", then we denoted this implication by "$p \Rightarrow q$".

then, "if p then q" is the same as the following :

(i) If both p and q are true, then $p \Rightarrow q$ is also true.

(ii) If p is true and q is false, then $p \Rightarrow q$ is false.

(iii) If p is false and q is true, then $p \Rightarrow q$ is true

(iv) If p and q both are false, then $p \Rightarrow q$ is true.

Consider the following statements

If a number is a multiple if 9, then it is a multiple of 3.

It is an implication having antecedent(p) and consequent(q) as :

p : a number is multiple of 9

q : a number is multiple of 3.

the above statement says that

(i) p is sufficient condition for q.

this says that knowing that a number is a multiple of 9 is sufficient to conclude that it is a multiple of 3.

(ii) p only if q.

This states that a number is a multiple of 9 only if it is a multiple of 3.

(iii) q is necessary condition for p.

This says that when a number is a multiple of 9, it is necessarily a multiple of 3.

(iv) $\sim q$ implies $\sim p$.

This says that if a number is not a multiple of 3, then it is not a multiple of 9.

38.6 CONTRAPOSITIVE AND CONVERSE

Contrapositive : If p and q are two statements, then the contrapositive of the implication "if p then q" is "if $\sim q$, then $\sim p$".

Converse : If p and q are two statements, then the converse of the implication "if p-then q" is "if q-then p".

For example,

If a number is divisible by 9, then it is divisible by 3.

Its implication is as follows:

p : number is divisible by 9.

q : a number is divisible by 3.

The contrapositive of this statement is

If a number is not divisible by 3, it is not divisible by 9.

The converse of the statement is

If a number is divisible by 3, then it is divisible by 9.

38.7 IF AND ONLY IF IMPLICATION

If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and it is denoted by $p \Leftrightarrow q$.

For example,

A triangle is equilateral if and only if it is equiangular.

This is if and only if implication with the component statements

p : A triangle is equilateral

q : A triangle is equiangular

Example 38.5 Write the following statements in the form "if then".

(i) You get job implies that your credentials are good.

(ii) The banana trees will bloom if it stays warm for a month.

(iii) A quadrilateral is a parallelogram if its diagonals bisect each other.

Solution : (i) We know that "if p-then q" is equivalent to "$p \Rightarrow q$".

Then the given statement can be written as

"If you get a job, then your credentials are good".

(ii) We know that "if p-then q" is equivalent to "$p \Rightarrow q$"

The given statement can be written as

"If it stays warm for a month, then the banana trees will bloom".

(iii) The given statement can be written as

"If the diagonals of a quadrilateral bisect each other, then it is a parallelogram"

Example 38.6 Write the contrapositive of the following statements :

(i) If a triangle is equilateral, it is isosceles.

(ii) If you are born in India, then you are a citizen of India.

(iii) x is an even number implies that x is divisible by 4.

Solution : The contrapositive of these statements are

(i) If a triangle is not isosceles, then it is not equilateral.

(ii) If you are not a citizen of India, then you are not born in India.

(iii) If x is not divisible by 4, then x is not an even number.

Example 38.7 Write the converse of the following statements :

(i) If a number n is even, then n^2 is even.

(ii) If x is even number, then x is divisible by 4.

Solution : The converse of these statements are :

(i) If a number n^2 is even, then n is even.

(ii) If x is divisible by 4, then x is even.

Example 38.8 Given below are two pairs of statements. Combine these two statements using "if and only if".

(i) p : if a rectangle is a square, then all its four sides are equal.

q : if all the four sides of a rectangle are equal, then the rectangle is a square.

(ii) p : if the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q : if a number is divisible by 3, then the sum of its digits is divisible by 3.

Solution : (i) A rectangle is a square if and only if all its four sides are equal.

(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

CHECK YOUR PROGRESS 38.2

1. Rewrite the following statement with "if-then" in five different ways conveying the same meaning.

   If a natural number is odd, then its square is also odd.

2. Write the contrapositive and converse of the following statements.

   (i) If you live in Kanpur, then you have winter clothes.

   (ii) If x is a prime number, then x is odd.

   (iii) If two lines are parallel, then they do not intersect in the same plane.

   (iv) x is an even number implies that x is divisible by 4.

   (v) Something is cold implies that it has low temperature.

3. Write each of the following statements in the form of "if-then".

   (i) To get an A^+ in the class, it is necessary that you do all the exercises of the book.

   (ii) The game is cancelled only if it is raining.

   (iii) It never rains when it is cold.

4. Rewrite each of the following statements in the form "if and only if".

   (i) If you watch television, then your mind is free and if your mind is free, then you watch television.

(ii) For you to get an A grade, it is necessary and sufficient that you do all the homework regularly.

38.8 VALIDATING STATEMENTS

In this section, we will discuss validity of statement. Checking the validity of statement means when it is true and when it is not true. The answer to these questions depend upon which of the special words and phrases "and", "or" and which of the implications "if and only if" "if-then", and which of the quantifiers "for every", "there exists", appear in the given statement.

Here, we shall discuss some techniques or rules to find when a statement is valid or true.

Rule 1 : Statements with "And"

If p and q are mathematical statements, then in order to show that the statement "p and q" is true, we follows the following steps :

Step-1 : Show that the statement p is true.

Step-2 : Show that the statement q is true.

Rule 2 : Statements with "or"

If p and q are mathematical statements, then in order to show that the statement "p or q" is true, one must consider the following.

Case 1 : Assuming that p is false, show that q must be true.

Case 2 : Assuming that q is false, show that p must be true.

Rule 3 : Validity of statements with "if-then".

If p and q are two mathematical statements, then to prove the statement "if p then q", we need to show that any one of the following case is true.

Case 1 : (Direct method)

By assuming that p is true, prove that q must true.

Case 2 : (Contrapositive method)

By assuming that q is false, prove that p must be false.

Rule 4 : Statements with "if and only if".

In order to prove the validity of the statement "p if and only if q" we need to show :

(i) If p is true then q is true.

(ii) If q is true the p is true.

Example 38.9 If p and q are two statements given by

p : 35 is multiple of 5

q : 35 is multiple of 6

Write the compound statement connecting these two statements with "and" and check the validity.

Solution : The compound statement "35 is multiple of 5 and 6. Since 35 is multiple of 5 but it is not multiple of 6. Therefore p is true but q is not true.

Example 38.10 Given below are two statements :

p : 35 is a multiple of 5

q : 35 is a multiple of 6

Write the compound statement connecting these two statements with "OR" and check its validity.

Solution : The compound statement is "35 is a multiple of 5 or 6."

By assuming that the statement q is false, then p is true.

Hence the compound statement is true i.e. valid.

Example 38.11 Check whether the following statement is true or not.

"If x and y are odd integers, then xy is an odd integer".

Solution : Let p and q be the statements given by

p : x and y are odd integers

q : xy is an odd integer

Then the given statement is

If p-then q.

Direct method : Let p be true, then,

p is true

$\Rightarrow$ x and y are odd integers

$\Rightarrow$ $x = 2m + 1,\ y = 2n + 1$ for some integers m, n

$\Rightarrow$ $xy = (2m + 1)\ (2n + 1)$

$\Rightarrow$ $xy = 2\ (2mn + m + n) + 1$

$\Rightarrow$ xy is an odd integer

$\Rightarrow$ q is true

Thus p is true $\Rightarrow$ q is true

Hence " if p-then q" is a true statement.

38.8.1 Contrapositive Method

Let q be not true. Then q is not true

$\Rightarrow$ xy is an even integer

$\Rightarrow$ either x is even or y is even or both x and y are even

$\Rightarrow$ p is not true

Thus q is false

$\Rightarrow$ p is false

Hence "If p-then q" is a true statement.

38.8.2 Validity of Statements by Contradiction

Here to check whether a statement p is true, we assume that p is not true i.e. $\sim p$ is true. Then we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true.

Example 38.12 Verify by the method of contradiction p : $\sqrt{7}$ is irrational.

Solution : Let p be the statement given by p : $\sqrt{7}$ is irrational.

We assume that $\sqrt{7}$ is rational

$\Rightarrow$ $\sqrt{7} = \dfrac{a}{b}$, where a and b are integers having no common factor.

$\Rightarrow$ $7 = \dfrac{a^2}{b^2}$

$\Rightarrow$ $a^2 = 7b^2$

$\Rightarrow$ 7 divides a^2

$\Rightarrow$ 7 divides a

$\Rightarrow$ $a = 7c$ for some integer c

$\Rightarrow$ $a^2 = 49c^2$

$\Rightarrow$ $7b^2 = 49c^2$

$\Rightarrow$ $b^2 = 7c^2$

$\Rightarrow$ 7 divides b^2

$\Rightarrow$ 7 divides b

Thus, 7 is common factor of both a and b. This contradicts that a and b have no common factor. So, our assumption $\sqrt{7}$ is rational is wrong. Hence the statement "$\sqrt{7}$ is irrational", is true.

CHECK YOUR PROGRESS 38.3

1. Check the validity of the following statements :
   (i) p : 80 is a multiple of 4 and 5.
   (ii) q : 115 is a multiple of 5 and 7.
   (iii) r : 60 is a multiple of 2 and 3.
2. Show that the statement

   p : "if x is a real number such that

   $x^3 + 2x = 0$, then x is 0" is true by (i) direct method (ii) method of contradiction (iii) method of contrapositive.

3. Show that the following statement is true by the method of contrapositive

   p : "if x is an integer and x^2 is odd x is also odd".

4. Show that the following statement is true.

   "The integer x is even if and only if x^2 is even.

5. Which of the following statements are true and which are false? In each case give a valid reason for saying so :

   (i) p : Each radius of a circle is a chord of the circle.

   (ii) q : The centre of a circle bisect each other chord of the circle.

   (iii) r : Circle is a particular case of an ellipse.

   (iv) s : If x and y are integers such that $x > y$, then $-x < -y$.

   (v) t : $\sqrt{11}$ is a rational number.

SUPPORTIVE WEB SITES

http://www.cs.odu.edu/~toida/nerzic/content/set/math_reasoning.html
http://www.freencertsolutions.com/mathematical-reasoning
www.basic-**mathematics**.com/examples-of-inductive-**reasoning**.html

TERMINAL EXERCISE

1. Write four examples of sentences which are not statements.
2. Are the following pairs of statements negations of each other :
   (i) The number x is not a rational number.
   The number x is not an irrational number.
   (ii) The number x is a rational number.
   The number x is an irrational number.
3. Write the contrapositive and converse of the following statements :
   (i) If two lines are parallel, then they donnot intersect in the same plane.
   (ii) If x is a prime number, then x is odd.
4. By giving a counter example, show that the following statements are not true :
   (i) p : if all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.
   (ii) q : the equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.
5. Let, p : 25 is a multiple of 5.
   q : 25 is a multiple of 8, be two statements.

   Write the compound statements with "And" and "or". In both the cases check the validity of the compound statements.

ANSWERS

CHECK YOUR PROGRESS 38.1

1. (i) Statements are, (ii), (iii), (iv), (vi), (vii), (viii), (xii)

CHECK YOUR PROGRESS 38.2

1. (i) $p \Rightarrow q$ i.e., n is an odd natural number $\Rightarrow x^2$ is an odd natural number.

   (ii) p is a sufficient condition of q.

   (iii) p only if q i.e, a natural number is odd only if its square is odd.

   (iv) q is necessary condition of p.

   (v) $\sim q \Rightarrow \sim p$ i.e., if the square of a natural number is not odd, then the natural number is not odd.

2. (i) Contrapositive : If you do not have winter clothes, then you do not live in Kanpur.

   Converse : If you have winter clothes, then you live in Kanpur.

   (ii) Contrapositive : If a number x is not odd, then x is not prime.

   Converse : If a number x is odd, then x is a prime number.

   (iii) Contrapositive : If two lines do not intersect in the same plane, then they are not parallel.

   Converse : If two lines do not intersect in the same plane, then they are parallel.

   (iv) Contrapositive : If x is not divisible by 4, then x is not an even number.

   Converse : If x is divisible by 4, then x is an even number.

   (v) Contrapositive : If something does not have low temperature, then it is not cold.

   Converse : If it has low temperature then something is cold.

3. (i) "If you get A^+ in the class, then you do all the exercise of the book."

   (ii) If it is raining , then the game is cancelled.

   (iii) If it is cold, then it never rains.

4. (i) You watch television if and only if your mind is free.

   (ii) You get an A grade if and only if you do all the homework regularly.

CHECK YOUR PROGRESS 38.3

1. (i) True (ii) False (iii) True

5. (i) False (ii) False (iii) True (iv) True (v) False.

MODULE - X
Linear Programming and Mathematical Reasoning

Notes

TERMINAL EXERCISE

1. (i) Everyone in this room is bald.

   (ii) "$\cos^2\theta$ is always greater than $\frac{1}{2}$."

   (iii) Mathematics is difficult.

   (iv) Listen to me, Sohan!

2. (i) Yes (ii) Yes

3. (i) Contrapositive : If two lines intersect in the same plane, then they are not parallel.

   Converse : If two lines do not intersect in the same plane, then they are parallel.

   (ii) Contrapositive : If a number x is not odd, then x is not a prime number.

   Converse : If a number x is odd, then it is a prime number.

5. The compound statement with "And" : 25 is a multiple of 5 and 8, which is a false statement.

   The compound statement with "or" : 25 is a multiple of 5 or 8. This is a true statements.

QUESTION PAPER DESIGN

Subject: Mathematics (311)

Senior Secondary Course

Maximum Marks: 100 **Time: 3 Hrs.**

1. Weightage by Objectives:

S. No.	Objectives	Marks	% of Total Marks
1.	Knowledge	30	30%
2.	Understanding	40	40%
3.	Application	22	22%
4.	Skill	08	8%

2. Module-wise Time and Mark Distribution:

S.No.	Type of question	No. of questions	Marks	Estimated time (in minutes)
1.	Long Answer (LA) (6 mark question)	5	30	$5 \times 10 = 50$
2.	Short Answer (SA) (4 mark question)	12	48	$12 \times 6 = 72$
3.	Very Short Answer (2 mark question)	6	12	$6 \times 3 = 18$
4.	MCQ (LA) (6 mark question)	10	10	$10 \times 2 = 20$
	Total	**33**	**100**	**160 Minutes**

**20 minutes for revision*

3. Weightage by Content:

S.No.	Module	No. of Lesson	Marks
1.	Algebra-II	03	17
2.	Relations and Functions	02	12
3.	Calculus	08	45
4.	Vectors and Three Dimensional Geometry	04	17
5.	Linear Programming and Mathematical Reasoning	02	09
	Total	**19**	**100**

4. Weightage by Difficulty Level:

Estimated Level	Marks	Percentage of marks
Difficulty	20	20
Average	50	50
Easy	30	30
Total	**100**	**100**

Sample Question paper

MATHEMATICS (311)

Maximum Marks : 100 Time : 3 Hours

Instructions:

(i) This question paper consists of four sections A,B,C and D.

(ii) Question number 1 to 10 in sections A are multiple choice questions. Each question carries one mark. In each question there are Four choices A,B,C,D, of which only one is correct you have to select the correct choice and indicate it in your answer book by writing (A), (B), (C) or (D) as the case may be.

(iii) Question number 11 to 16 in sections B are very short answer questions and carry 2 marks each.

(iv) Question number 17 to 28 in section C are short answer questions and carry 4 marks each.

(v) Question number 29 to 33 in section D are long answer questions and carry 6 marks each.

(vi) All questions are compulsory. There is no overall choice however alternative choices are given in some questions. In such question you have to attempt only one choice.

Section-A

1. Let A be a square matrix of order 3 x 3 , then $|KA|$ is equal to:

   (A) $k|A|$ (B) $3k|A|$ (C) $k^2|A|$ (D) $k^3|A|$

2. If $\tan^{-1} x = y, \ x \in R$ then

   (A) $0 \le y \le \pi$ (B) $0 < y < \pi$ (C) $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ (D) $\frac{-\pi}{2} < y < \frac{\pi}{2}$

3. The distance of plane $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 3$, from origin is :

   (A) 3 (B) $\sqrt{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) 0

4. Which of the following sentences is not a statement ?

   (A) 5 is greater then 12

   (B) Every set is a finite set

   (C) The sun is a star

   (D) How far is Agra from here ?

5. Let R $= \{(1,2), (2,2), (1,1), (4,4), (1,3), (3,3), (3,2)\}$ be a relation on the set $\{1, 2, 3, 4,\}$, then

   (A) R is reflexive and symmetric but not transitive

   (B) R is symmetric and transitive but not reflexive

   (C) R is reflexive and transtive but not symmetric

   (D) R is an equivalence relation .

6. The values of x for which , $f(x) = |x| + |x+5| + |x-6|$ is not differentiable are :

   (A) 0,5,6 (B) 0,-5,-6 (C) 0,-5, 6 (D) 0,5,-6

7. If $y = \log\left(x.e^x\right)$, then $\frac{dy}{dx}$ is :

   (A) $\frac{x+1}{x}$ (B) $\frac{x+1}{x.e^x}$ (C) $e^x\left(x+1\right)$ (D) $\frac{1}{xe^x}$

8. If $\int e^x\left(\cos ec^2x - \cot x\right)dx = p.e^x + c$, then p is :

   (A) $\cos ec^2 x$ (B) $\cot x$ (C) $-\cot x$ (D) $\cos ecx.\cot x$

9. The value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} |\sin x|\, dx$ is

   (A) -2 (B) 0 (C) 1 (D) 2.

10. degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + 3\left(\frac{dy}{dx}\right)^3 + 4y = 0$ is :

    (A) 3 (B) 2 (C) 1 (D) not defined .

Section-B

11. If $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$, then find X and Y .

    OR

    Construct a 2x2 matrix A whose elements in the ith row and j^{th} column are given by $a_{ij} = \frac{3i - j}{2}$.

12. Let f : $R \to R$ and g : $R \to R$ be defind by f $(x) = x+1$ and g $(x) = x-1$, then show that fog = gof .

13. Evaluate :

    $$\lim_{x \to 0.} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$$

14. If $y = \sin^{-1} x$, show that $\frac{d^2y}{dx^2} = \frac{x}{\left(1-x^2\right)^{3/2}}$.

15. Find the area of a parallelogram whose adjacent sides are given by $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and

    $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

16. Check whether the following statement is true or not

    If x,y $\in Z$ are such that x and y are odd , then x y is odd.

Section-C

17. Express the following matrix as the sum of a symmetric and a skew symmetric matrix .

$$\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$$

18. If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, then find the following :

(i) $\vec{a} + \vec{b}$ (ii) $\vec{a} - \vec{b}$

(iii) $(\vec{a} + \vec{b}).(\vec{a} - \vec{b})$ (iv) angle between $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

19. Using properties of determinants prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

OR

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and $A^2 + kA - 5I = 0$ Where k is any real number, then find the value of k

20. Prove : $\tan^{-1}\left(\frac{5}{12}\right) + \cos ec^{-1}\left(5\sqrt{2}\right) + \tan^{-1}\left(\frac{16}{63}\right) = \frac{\pi}{4}$

21. Let f : $R \to R$ be defiend as f (x) = 4x + 3. show that f is one - one and on to . Hence find the inverse of function f.

22. Find the values of a and b such that the function given by

$$f(x) = \begin{Bmatrix} 5 \; if \;\; x \le 2 \\ ax + b \; if \;\; 2 < x < 10 \\ 21 \; if \;\; x \ge 10 \end{Bmatrix}$$, is a continuous function .

23. If $y = x^{\cos x} + \frac{x^2 + 1}{x^2 - 1}, \; find \; \frac{dy}{dx}$

24. Find the intervals in which the function given by f $(x) = -2x^3 - 9x^2 - 12x + 1$ is (i) increasing (ii) decreasing.

OR

Find the equation of tangent to the curve $y = x^2 + 4x + 1$ at $x = 3$. Also find the point where the tangent to the curve is parallel to x- axis.

25. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt{\tan x}} dx$

26. Solve the differential equation : $(x-y)\frac{dy}{dx} = x+3y.$

27. The magnitide of vector product of the vector $\hat{i}+\hat{j}+\hat{k}$ with the sum of the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to $2\sqrt{26}$. Find the value of λ.

28. Find : $\int \frac{dx}{\sqrt{(x-1)(2x-3)}}$

OR

Find : $\int \frac{3x+2}{(x-1)(2x+3)} dx.$

Section-D

29. Solve the following system of linear equations using matrix method :

$x-y+2z=7, 3x+4y-5z=-5, 2x-y+3z=12.$

OR

Find the inverse of the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ using elementary transformation method.

30. Show that of all the rectangles inscribed in a given circle, the square has maximum area.

OR

A rectangular sheet of tin 45cm by 24cm is to be made into a box without top by cutting off squares from each corner and folding up flaps. what should be the side of the square to be cut off so that the volume of the box is maximum?

31. Using integration, find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1.$

32. Find the equation of the plane passing through the point $(1, 2, -4)$ and parallel to the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6} \text{ and } \frac{x-1}{1}=\frac{y+3}{1}=\frac{z-5}{-1}$$

33. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts . He earns a profit of ₹ 20 per package on nuts and ₹ 10 per package on bolts. How many packages of each should be produced each day so as to maximize his profit if he operates his michines for at most 12 hours a day ? Form the above as a linear programming problem and solve it graphically.

Marking Scheme

Q.No.	Value Points	Marks Distribution	Total marks
1.	D.		1
2.	D		1
3.	B		1
4.	D		1
5.	C		1
6.	C		1
7.	A		1
8.	C		1
9.	D		1
10.	B		1
11.	$X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$	1	
	$Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$	1	
	or		
	$A = \begin{bmatrix} 1 & 1/2 \\ 5/2 & 2 \end{bmatrix}$	$\frac{1}{2}$ mark for each correct element	2
12.	$\text{fog}\,(x) = f\left(g(x)\right) = f\,(x-1) = x-1+1 = x$	1	
	$\text{gof}\,(x) = g\left(f(x)\right) = g\,(x+1) = x+1-1 = x$	1	2
13.	$\lim\limits_{x\to 10} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} \times \dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$	$\frac{1}{2}$	
	$= \lim\limits_{x\to 10} \dfrac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim\limits_{x\to 10} \dfrac{2}{\sqrt{1+x}+\sqrt{1-x}}$	1	
	$= \dfrac{2}{1+1} = 1$	$\frac{1}{2}$	2

14.	$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}=\left(1-x^2\right)^{-1/2}$ $\frac{d^2y}{dx^2}=-\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}-1}(-2x)$ $=\frac{x}{\left(1-x^2\right)^{3/2}}$	1 1	 2
15.	$\vec{a}\times\vec{b}=\begin{vmatrix} i & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & 2 & 3 \end{vmatrix}$ $=-22\hat{i}-\hat{j}+8\hat{k}$ $\left\|\vec{a}\times\vec{b}\right\|=\sqrt{(-22)^2+(-1)^2+(8)^2}$ $=\sqrt{549}=3\sqrt{61}$ $\therefore$ Area of parallelogram $=3\sqrt{61}\ unit^2$	 1 $\frac{1}{2}$ $\frac{1}{2}$	 2.
16.	Let p: x . y $\in$ Z such that x and y are odd q: xy is odd. we assume that if p is true, then q is true. P is true means, let x = 2m +1, y= 2n+1 where m,n are integers $\therefore \quad xy=(2m+1)(2n+1)$ $=2(2mn+m+n)+1$ This shows that xy is odd i.e q is true	$\frac{1}{2}$ $\frac{1}{2}$ 1	 2.
17.	Let $A=\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ $\therefore A'=\begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$ $A+A'=\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}$	1 $\frac{1}{2}$	

	$\dfrac{A+A^1}{2}=\begin{bmatrix}1 & -3/2 & 1/2\\ -3/2 & 8 & 9/2\\ 1/2 & 9/2 & 5\end{bmatrix}$	$\frac{1}{2}$	
	$A-A^1=\begin{bmatrix}0 & 9 & 9\\ -9 & 0 & -3\\ -9 & 3 & 0\end{bmatrix}$	$\frac{1}{2}$	
	$\dfrac{A-A^1}{2}=\begin{bmatrix}0 & 9/2 & 9/2\\ -9/2 & 0 & -3/2\\ -9/2 & 3/2 & 0\end{bmatrix}$	$\frac{1}{2}$	
	We know that $\dfrac{A+A^1}{2}$ is symmetric and $\dfrac{A-A^1}{2}$ is skew symmetric matrix	$\frac{1}{2}$	
	$\therefore A=\begin{bmatrix}1 & -3/2 & 1/2\\ -3/2 & 8 & 9/2\\ 1/2 & 9/2 & 5\end{bmatrix}+\begin{bmatrix}0 & 9/2 & 9/2\\ -9/2 & 0 & -3/2\\ -9/2 & 3/2 & 0\end{bmatrix}$	$\frac{1}{2}$	4
18.	(i) $\vec{a}+\vec{b}=4\hat{i}+\hat{j}-\hat{k}.$	1	
	(ii) $\vec{a}+\vec{b}=-2\hat{i}+3\hat{j}-5\hat{k}$	1	
	(iii) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=-8+3+5=0$	1	
	(iv) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$		
	$\Rightarrow \vec{a}+\vec{b}$ is perpendicular to $\vec{a}-\vec{b}$	1	4
19.	$LHS=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$		
	Applying $R_1\to R_1-R_2$ & $R_2\to R_2-R_3$		

$LHS = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$	$1\frac{1}{2}$	
Taking (a-b) common from R_1 and (b-c) common from R_2 we get		
LHS = (a-b) (b-c) $\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$	$1\frac{1}{2}$	
Now expanding using c_1		
LHS = (a-b) (b-c)$\left[0-0+1(b+c-a-b)\right]$		
= (a - b) (b - c) (c - a) = RHS	1	
OR		
$A^2 = A.A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$	1	
kA = k $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} k & 2k & 2k \\ 2k & k & 2k \\ 2k & 2k & k \end{bmatrix}$	1	
51 = 5 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$	$\frac{1}{2}$	
Now $A^2 + kA - 5I = 0$		
$\Rightarrow \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} k & 2k & 2k \\ 2k & k & 2k \\ 2k & 2k & k \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
$\Rightarrow \begin{bmatrix} 4+k & 8+2k & 8+2k \\ 8+2k & 4+k & 8+2k \\ 8+2k & 8+2k & 4+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	1	
$\Rightarrow 4+k=0$ i.e k = - 4 .	$\frac{1}{2}$	4

20.	$\operatorname{cosec}^{1}\left(5\sqrt{2}\right)=\tan^{-1}\left(\frac{1}{7}\right)$	1	
	$\therefore \quad \text{LHS}=\tan^{-1}\left(\frac{5}{12}\right)+\operatorname{cosec}^{-1}\left(5\sqrt{2}\right)+\tan^{-1}\left(\frac{16}{63}\right)$		
	$=\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{16}{63}\right)$		
	$=\tan^{-1}\left[\frac{\frac{5}{12}+\frac{1}{7}}{1-\frac{5}{12}.\frac{1}{7}}\right]+\tan^{-1}\left(\frac{16}{63}\right)$	1	
	$=\tan^{-1}\left(\frac{47}{79}\right)+\tan^{-1}\left(\frac{16}{63}\right)$	$\frac{1}{2}$	
	$=\tan^{-1}\left[\frac{\frac{47}{79}+\frac{16}{63}}{1-\frac{47}{79}.\frac{16}{63}}\right]$	1	
	$=\tan^{-1}(1)=\frac{\pi}{4}=RHS$	$\frac{1}{2}$	4
21.	(i) Let x_1, x_2 be any two elements of domain such that $f(x_1)=f(x_2)$ $\Rightarrow 4x_1+3=4x_2+3$ $\Rightarrow x_1=x_2$ $\therefore$ f is one- one function.	$1\frac{1}{2}$	
	(ii) Let y be any element of codomain such that f (x) = y $\Rightarrow 4x+3=y, \Rightarrow x=\frac{y-3}{4}$ clearly for every $y \in$ codmain there exists $x \in$ domain $\therefore$ every y $\in$ codomain has pre image $x=\frac{y-3}{4} \in$ domain $\therefore$ f is on -to function	$1\frac{1}{2}$	

	(iii) since f is one - one and on to and hence it is invertible. $\therefore\ f^{-1}: R \to R$ exists and it is defined by $f^{-1}(y) = \dfrac{y-3}{4}$	1	4
22.	$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} 5 = 5$ $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (ax+b) = 2a+b.$ since f is a continuous function $\therefore\quad 2a + b = 5 \quad ...(i)$	$1\frac{1}{2}$	
	Now $\lim\limits_{x\to 10^-} f(x) = \lim\limits_{x\to 10^-} (ax+b) = 10a+b$ $\lim\limits_{x\to 10^+} f(x) = \lim\limits_{x\to 10^+} 21 = 21$ Again f is continuous function $\therefore\quad 10a + b = 21 \quad ...(ii)$	$1\frac{1}{2}$	
	Solving (i) and (ii) to get a = 2 and b = 1	1	4
23.	$y = x^{\cos x} + \dfrac{x^2+1}{x^2-1}$ $\therefore\quad \dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^{\cos x}\right) + \dfrac{d}{dx}\left(\dfrac{x^2+1}{x^2-1}\right) \quad ...(i)$	$\frac{1}{2}$	
	Let u = $x^{\cos x}$ $\therefore$ log u = cos x . log x $\dfrac{1}{u}\cdot\dfrac{du}{dx} = \cos x.\dfrac{1}{x} + \log x(-\sin x)$ $\Rightarrow\quad \dfrac{du}{dx} = x^{\cos x}\left[\dfrac{\cos x}{x} - \sin x.\log x\right] \quad ...(ii)$	2	
	$\dfrac{d}{dx}\left[\dfrac{x^2+1}{x^2-1}\right]$ $= \dfrac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2} =$ $= \dfrac{4x}{(x^2-1)^2} \quad ...(iii)$	1	

	From (i), (ii) and (iii) we get $\frac{dy}{dx}=x^{\cos x}\left[\frac{\cos x}{x}-\sin x.\log x\right]-\frac{4x}{\left(x^2-1\right)^2}.$	$\frac{1}{2}$	4
24.	$f'(x)=-6x^2-18x-12$ $=-6\left(x^2+3x+2\right)=-6(x+2)(x+1)$	$\frac{1}{2}$	
	For increasing and decreasing function $f'(x)=0 \Rightarrow x=-2,-1.$	$\frac{1}{2}$	
	$\therefore$ Intervals are $(-\infty,-2],(-2,-1],[-1,\infty)$	1	

Interval	Sign of $f'(x)$	Conclusion
$(-\infty,-2]$	$(-)(-)(-)=-ve$	f is decreasing
$[-2,-1]$	$(-)(+)(-)=+ve$	f is increasing
$[-1,\infty)$	$(-)(+)(+)=-ve$	f is decreasing

$1\frac{1}{2}$

$\therefore$ f is increasing is $[-2,-1]$ and it is decreasing in $(-\infty,-2]\cup[-1,\infty)$ — $\frac{1}{2}$

OR

when x = 3, y = 22. — $\frac{1}{2}$

$\frac{dy}{dx}=2x+4.$ — $\frac{1}{2}$

$\frac{dy}{dx}\, at\, x=3=10.$

$\therefore$ Equation of tangent is — $\frac{1}{2}$

$(y-22)=10(x-3)$

$\Rightarrow 10x-y=8.$ — 1

For tangent to be parallel to x - axis

$f'(x)=0$ — $\frac{1}{2}$

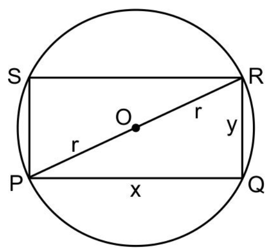
	$\Rightarrow \quad 2x+4=0$ $\Rightarrow \quad x=-2.$ When x = -2, y = -3. $\therefore$ Required point is $(-2,-3)$	$\frac{1}{2}$ $\frac{1}{2}$	4
25.	$I=\int_0^{\pi/2}\frac{dx}{1+\sqrt{\tan x}},\ I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad ...(i)$ $I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x+\sqrt{\cos x}}}dx, \quad ...(ii)$ $\left(\therefore \int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$ Adding (i) and (ii) we get $2I=\int_0^{\pi/2}\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx=\int_0^{\pi/2}1.dx$ $=[x]_0^{\pi/2}=\frac{\pi}{2}-0=\frac{\pi}{2}$ $\therefore \quad I=\frac{\pi}{4}$	1 1 1 1	4
26.	The given differential equation can be written as $\frac{dy}{dx}=\frac{x+3y}{x-3y} \quad ...(i)$ This is a homogeneous differential equation $\therefore$ substituting $y=vx$ $\frac{dy}{dx}=v+x.\frac{dv}{dx}$ $\therefore (i)$ becomes $v+x\frac{dv}{dx}=\frac{x+3vx}{x-vx}$ $\Rightarrow \quad v+x\frac{dv}{dx}=\frac{1+3v}{1-v}$	$\frac{1}{2}$ $\frac{1}{2}$	

	$\Rightarrow \quad x\dfrac{dv}{dx}=\dfrac{1+3v}{1-v}-v \;=\dfrac{(1+v)^2}{1-v}$		
	$\Rightarrow \quad \dfrac{1-v}{(1+v)^2}dv=\dfrac{dx}{x}$	1	
	$\therefore \quad \displaystyle\int\dfrac{1-v}{(1+v)^2}dv=\int\dfrac{dx}{x}$		
	$\Rightarrow \quad 2\displaystyle\int\dfrac{1}{(1+v)2}dv-\int\dfrac{1}{(1+v)}dv=\log x+c_1$		
	$\Rightarrow \quad \dfrac{-2}{1+v}-\log\lvert .1+v\rvert=\log x+c_1$	1	
	$\dfrac{-2x}{x+y}-\log\left\lvert\dfrac{x+y}{x}\right\rvert-\log x=c_1$		
	$\dfrac{-2x}{x+y}-\left[\log\left\lvert\dfrac{x+y}{x}.x\right\rvert\right]=c$		
	$\Rightarrow \quad \dfrac{2x}{x+y}+\log\lvert x+y\rvert=c_1\left(where_{c_1=-c}\right)$	1	4
27.	Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$		
	$\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$		
	$\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$		
	$\therefore \quad \vec{b}+\vec{c}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}.$	$\dfrac{1}{2}$	
	$\vec{a}\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 2+\lambda & 6 & -2\end{vmatrix}$		
	$=\hat{i}(-2-6)-\hat{j}(-2-2-\lambda)+\hat{k}(6-2-\lambda)$		
	$=-8\hat{i}+(4+\lambda)\hat{y}+(4-\lambda)\hat{k}.$	1	
	$\lvert\vec{a}\times(\vec{b}+\vec{c})\rvert \;=\sqrt{(-8)^2+(4+x)^2+(4-\lambda)^2}$		

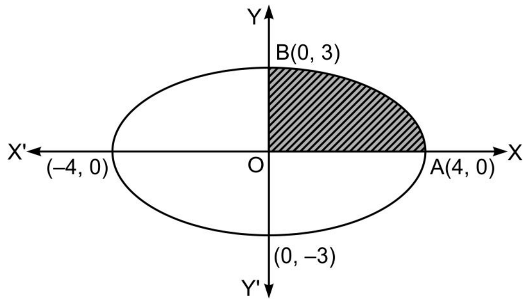
	$=\sqrt{2\lambda^2+96}.$	1	
	Now $\sqrt{2\lambda^2+96}=2\sqrt{26}.$		
	$\Rightarrow \sqrt{2\lambda^2+96=}$ 104		
	$\Rightarrow \quad 2\lambda^2=8$		
	$\Rightarrow \quad \lambda^2=4$	1	
	$\Rightarrow \quad \lambda=\pm 2$	$\frac{1}{2}$	4
28.	$I=\int \frac{1}{\sqrt{2x^2-5x+3}}dx$		
	$=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2-\frac{5}{2}x+\frac{3}{2}}}dx$	$\frac{1}{2}$	
	$=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x-5/4)^2-(1/4)^2}}dx.$	$1\frac{1}{2}$	
	$=\frac{1}{\sqrt{2}}.\log\left\|\left(x-\frac{5}{4}\right)+\sqrt{x^2-\frac{5}{2}x+3/2}\right\|+c_1$	1	
	$=\frac{1}{\sqrt{2}}\log\left\|4x-5+2\sqrt{2}\sqrt{2x^2-5x+3}+C\right.$		
	where C = c_1 -log 4	1	
	OR		
	$I=\int \frac{3x+2}{(x-1)(2x+3)}$		
	Again let $\frac{3x+2}{(x-1)(2x+3)}=\frac{A}{x-1}+\frac{B}{2x+3}$	1	
	$\Rightarrow 3x+2=A(2x+3)+B(x-1)$		
	Putting x = - 3/2 we get B = 1		
	and x = 1, we get A = 1	1	
	$\therefore I=\int \frac{1}{x-1}dx+\int \frac{1}{2x+3}dx$		
	$=\log\|x-1\|+\frac{\log\|2x+3\|}{2}+c.$	2	4

29.	Let $A=\begin{bmatrix}1&-1&2\\3&4&-5\\2&-1&3\end{bmatrix}$ $x=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}7\\-5\\12\end{bmatrix}$		
	$\therefore \quad AX=B$		
	i.e $\quad X=A^{-1}B.$ (i)	1	
	$\lvert A\rvert=\begin{vmatrix}1&-1&2\\3&4&-5\\2&-1&3\end{vmatrix}=1(12-5)+1(9+10)+2(-3-8)$		
	$=7+19-22=4\neq 0.$		
	$\therefore A^{-1}$*exists*	1	
	Adj $A=\begin{bmatrix}7&1&-3\\-19&-1&11\\-11&-1&7\end{bmatrix}$	2	
	$\therefore A^{-1}=\frac{1}{4}\begin{bmatrix}7&1&-3\\-19&-1&11\\-11&-1&7\end{bmatrix}$	$\frac{1}{2}$	
	$\therefore \begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{4}\begin{bmatrix}7&1&-3\\-19&-1&11\\-11&-1&7\end{bmatrix}\begin{bmatrix}2\\1\\3\end{bmatrix}$		
	$=\frac{1}{4}\begin{bmatrix}8\\4\\12\end{bmatrix}=\begin{bmatrix}2\\1\\3\end{bmatrix}$	1	
	$\Rightarrow x=2, \quad y=1, \quad z=3,$	$\frac{1}{2}$	
	OR		
	Let $A=\begin{bmatrix}2&3&1\\2&8&1\\3&7&2\end{bmatrix}$		
	let $A=IA.$	$\frac{1}{2}$	

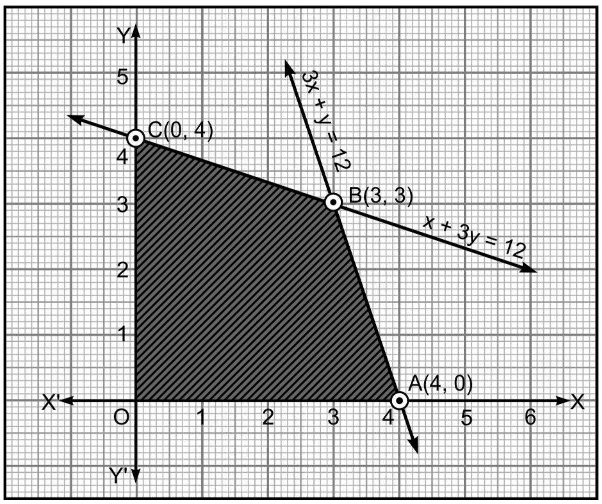
	i.e $\begin{bmatrix} 2 & 3 & 1 \\ 2 & 8 & 1 \\ 3 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$		
	Applying $R_1 \to \frac{1}{2} R_1$ $\begin{bmatrix} 1 & 3/2 & 1/2 \\ 2 & 8 & 1 \\ 3 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$	$\frac{1}{2}$	
	Applying $R_2 \to R_2 - 2R_1 and R_3 \to R_3 - 3R_1$ $\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 5 & 0 \\ 0 & 5/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} A.$	1	
	Applying $R_2 \to \frac{1}{5} R_2$ $\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 5/2 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{5} & \frac{1}{5} & 0 \\ -3/2 & 0 & 1 \end{bmatrix} A$	1	
	Applying $R_1 \to R_1 - \frac{3}{2} R_2$ and $R_3 \to R_3 - \frac{5}{2} R_2$ $\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 4/5 & -3/10 & 0 \\ -1/5 & 1/5 & 0 \\ -1 & \frac{-1}{2} & 1 \end{bmatrix} A$	1	
	Applying $R_3 \to 2R_3$. $\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4/5 & -3/10 & 0 \\ -1/5 & 1/5 & 0 \\ -2 & -1 & 2 \end{bmatrix} A$	1	

	Applying $R_1 \to R_1 - \frac{1}{2}R_3$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9/5 & 1/5 & -1 \\ -1/5 & 1/5 & 0 \\ -2 & -1 & 2 \end{bmatrix} A$$ Hence $A^{-1} = \begin{bmatrix} 9/5 & 1/5 & -1 \\ -1/5 & 1/5 & 0 \\ -2 & -1 & 2 \end{bmatrix}$	1	6
30.		corect figure $\frac{1}{2}$ mark	
	A (Area of rectangle) = x . y ...(i)	$\frac{1}{2}$	
	In Δ P Q R, $x^2 + y^2 = 4r^2$ $\Rightarrow \quad y = \sqrt{4r^2 - x^2}$ ...(ii)	$\frac{1}{2}$	
	$\therefore A = x\sqrt{4r^2 - x^2}$	$\frac{1}{2}$	
	let $Z = A^2 = x^2\left(4r^2 - x^2\right)$ i.e $Z = 4r^2x^2 - x^4$ $\frac{dZ}{dx} = 8r^2x - 4x^3$	1	
	For maxima or minima $8r^2x - 4x^3 = 0$ $4x\left(2r^2 - x^2\right) = 0$ $\Rightarrow x = 0 \quad or \quad x = \sqrt{2}.r.$ x = 0 is not possible	1	

	$\frac{d^2Z}{dx^2} = 8r^2 - 12x^2$		
	$\frac{d^2Z}{dx^2}$ is negative for $x = \sqrt{2}.r$		
	$\therefore Z = A^2$ is is maximum for $x = \sqrt{2}.r.$	1	
	From (ii) we get $y = \sqrt{4r^2 - 2r^2} = \sqrt{2}.r.$	$\frac{1}{2}$	
	Hence Area is maximum when x = y.	$\frac{1}{2}$	
	OR		
	Let x be the side of the square to be cut off from each corner of the sheet.		
	V(volume of box)		
	$= (45-2x)(24-2x)(x)$	1	
	$V = 4x^3 - 138x^2 + 1080x.$	$\frac{1}{2}$	
	$\frac{dV}{dx} = 12x^2 - 276x + 1080$	1	
	For maxima or minima		
	$\frac{dV}{dx} = 0$		
	$\Rightarrow 12x^2 - 276x + 1080 = 0$		
	$\Rightarrow x^2 - 23x + 90 = 0$	1	
	$\Rightarrow (x-18)(x-5) = 0$		
	$\Rightarrow x = 5 \; or \; x = 18(notpossible)$	1	
	$\frac{d^2V}{dx^2} = 24x - 276$		
	$\frac{d^2V}{dx^2}$ is negative for x = 5		
	$\therefore$ V is maximum for x = 5	1	
	Hence required side of square to be cut off is 5 cm.	$\frac{1}{2}$	6

31.		1	
	Required Area = 4 x Area 0AB	1	
	$= 4\int_0^4 \frac{3}{4}\sqrt{16-x^2}\,dx.$	1	
	$= 3\left[\frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$	1	
	$= 3\left[0 + 8\sin^{-1}(1) - 0 + 8\sin^{-1}(0)\right]$	1	
	$= 3\left[\frac{8\pi}{2}\right]$		
	$= \ 12\pi \ \ unit^2.$	1	6
32.	Let $a(x-1)+b(y-2)+c(z+4)=0$ (1) ...(i)	2	
	be the equation of required plane. Since plane is perpendicular to the given lines.		
	$\therefore 2a+3b+6c=0$ ...(ii)	1	
	a + b - c = 0 ...(iii)	1	
	$\frac{a}{-3-6} = \frac{b}{6+2} = \frac{c}{2-3}$		
	$\Rightarrow \frac{a}{-9} = \frac{b}{8} = \frac{c}{-1}$		
	$\frac{a}{9} = \frac{b}{-8} = \frac{c}{1} = \lambda(say)$		
	$\therefore a = 9\lambda, \quad b = -8\lambda, \ c = \lambda$	1	
	substituting the values of a, b, cin (i)		
	$9\lambda(x-1) - 8\lambda(y-2) + \lambda(z+4) = 0$		
	$\Rightarrow 9x - 8y + z + 11 = 0.$	1	6

33.	let x denotes the number of packages of nuts and y denotes the number of packages of bolts.		
	maximize , z = 20 x + 10 y	$\frac{1}{2}$	
	subject to the following constraints,		
	$x + 3y \leq 12$		
	$3x + y \leq 12$		
	$x >, 0, \quad y >, 0.$	$1\frac{1}{2}$	



	corner points of feasible region are		
	$O(0,0), A(4,0), B(3,3), C(0,4)$	$\frac{1}{2}$	
	Z at O (0,0), = 0 + 0 = 0		
	Z at $\text{A}(4,0) = 20 \times 4 + 10 \times 0 = 80$		
	Z at B(3,3) = $20 \times 3 + 10 \times 3 = 60 + 30 = 90.$		
	Z at (0,4) = $20 \times .0 + 10 \times 4 = 0 + 40 = 40$	1	
	Hence for maximum profit 3 packets of each		
	of nuts and bolts be produced	$\frac{1}{2}$	6.

APPENDIX-D

First Fold

Feed back on Lesson No.

Lesson No.	Lesson Name	Was the Content				Was the language		Were the Illustrations		What you have learnt is		
		Easy	Difficult	Interesting	Confusing	Simple	Complex	Useful	Not Useful	Very helpful	Somewhat helpful	Not Helpful

Final Fold and seal

Fourth Fold

Third Fold

Feed back on Questions

Lesson No.	Lesson Name	Intest Questions		Terminal Questions		
		Useful	Not useful	Easy	Diff.	V.diff.

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Thank you
Course Co-ordinator
Mathematics

Second Fold

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Did you consult any other book to study Mathematics? Yes/No

If yes, give reason for consulting it.

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Yours suggestion

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